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Introduction

The analytic method of Hardy and Littlewood (sometimes called the 'circle method') was developed for the treatment of *additive problems* in the theory of numbers. These are problems which concern the representation of a large number as a sum of numbers of some specified type. The number of summands may be either fixed or unrestricted; in the latter case we speak of *partition problems.* The most famous additive problem is Waring's problem, where the specified numbers are the kth powers, so that the problem is that of representing a large number N as

$$N = x_1^k + x_2^k + \cdots + x_s^k, \tag{1.1}$$

where s and k are given and $x_1, \ldots, x_s$ are positive integers. Almost equally famous is Goldbach's ternary problem, where the specified numbers are the primes, and the problem is that of representing a large number N as

$$N = p_1 + p_2 + p_3.$$

The great achievements of Hardy and Littlewood were followed later by further remarkable progress made by Vinogradov, and it is not without justice that our Russian colleagues now speak of the 'Hardy–Littlewood–Vinogradov method'.

It may be of interest to recall that the genesis of the Hardy–Littlewood method is to be found in a paper of Hardy and Ramanujan [69] in 1917 on the asymptotic behaviour of $p(n)$, the total number of partitions of n. The function $p(n)$ increases like $e^{A\sqrt{n}}$, where A is a certain positive constant; and Hardy and Ramanujan obtained for it an asymptotic series, which, if one stops at the smallest term, gives $p(n)$ with an error $O(n^{-1/4})$. The underlying explanation for this high degree of accuracy, which Hardy describes as 'uncanny', was given by Rademacher [68] in 1937: there is a convergent series which represents $p(n)$ exactly, and this is initially almost the same as the asymptotic series. There is one other group of problems in which the Hardy–Littlewood method leads to exact formulae; these are problems concerning the representation of a number as the sum of a given number of squares. It seems unlikely that there are any such formulae for higher powers.

Waring's problem is concerned with the particular Diophantine equation (1.1). There is no difficulty of principle in extending the Hardy–Littlewood method to deal with more general equations of additive type[1], say

$$N = f(x_1) + f(x_2) + \cdots + f(x_s),$$

where $f(x)$ is a polynomial taking integer values; in particular to the equation

$$N = a_1 x_1^k + a_2 x_2^k + \cdots + a_s x_s^k. \tag{1.2}$$

It is only in recent years, however, that much progress has been made in adapting the method to Diophantine equations of a general (that is, non-additive) character. An account of these developments will be given later in these lectures, but we shall be concerned at first mainly with Waring's problem and with additive equations of the type (1.2). All work on general Diophantine equations depends heavily on either the methods or the results of the work on additive equations.

Finally, we shall touch on the subject of Diophantine inequalities. Here, too, some results of a general character are now known, but they are less complete and less precise than those for equations.

[1] See the monograph [63].