8

The equation $c_1x_1^k + \cdots + c_sx_s^k = 0$

We now study the solutions of the above equation in integers, positive or negative, where $c_1, \ldots, c_s$ are fixed integers, none of them 0. If k is even, we must obviously suppose that not all the coefficients are of the same sign. If k is odd, we can ensure this by changing x_i into $-x_i$ if necessary. Hence, with a slight change of notation, we can write the equation as

$$c_1x_1^k + \cdots + c_rx_r^k - c_{r+1}x_{r+1}^k - \cdots - c_sx_s^k = 0, \tag{8.1}$$

where $c_1, \ldots, c_s$ are now positive integers and $1 \leq r \leq s-1$. We study the solutions of (8.1) in *positive* integers.

The first difference, in comparison with the equation treated in Chapter 7, is that there is no large number N which imposes restrictions on the sizes of the unknowns. We must therefore ourselves prescribe ranges for the variables, and the obvious way to do this is to choose a large number P, define $P_j = [P/c_j^{1/k}]$ for $1 \leq j \leq s$, and consider the number of solutions of (8.1) subject to

$$1 \leq x_j \leq P_j, \quad (1 \leq j \leq s). \tag{8.2}$$

We define the exponential sums $T_j(\alpha)$ as before, in (7.1). Then the number $\mathcal{N}(P)$ of solutions of (8.1), subject to (8.2), is given by

$$\mathcal{N}(P) = \int_0^1 T_1(\alpha) \cdots T_r(\alpha) T_{r+1}(-\alpha) \cdots T_s(-\alpha) d\alpha.$$

We follow again the treatment of Waring's problem, with the same slight changes as in the preceding section. The only further changes arise from the absence of N in the singular series and in the singular integral. In (4.10), we have to replace $J(P^\delta)$ by

$$(c_1 \cdots c_s)^{-1/k} \int_{|\gamma|<P^\delta} \left(\prod_{j=1}^{s} \int_0^1 e(\pm\gamma\xi_j^k) d\xi_j \right) d\gamma,$$

where the sign is $+$ for $j \le r$ and $-$ for $j > r$. As in (4.16) we are led to the evaluation of the integral

$$J = \int_{-\infty}^{\infty} k^{-s} \left(\prod_{j=1}^{s} \int_0^1 \zeta_j^{-1+1/k} e(\pm\gamma\zeta_j) d\zeta_j \right) d\gamma.$$

As in the proof of Theorem 4.1, we make a change of variable from ζ_s to u, where

$$\zeta_1 + \cdots + \zeta_r - \zeta_{r+1} - \cdots - \zeta_s = u,$$

and we find that

$$J = k^{-s} \int_0^1 \cdots \int_0^1 \{\zeta_1 \cdots \zeta_s(\zeta_1 \pm \cdots \pm \zeta_{s-1})\}^{-1+1/k}\, d\zeta_1 \cdots d\zeta_{s-1},$$

where $0 < \zeta_1 \pm \cdots \pm \zeta_{s-1} < 1$. All we need to know is that $J > 0$, and this is the case because there is some open set contained in $0 < \zeta_j < 1$ throughout which

$$0 < \zeta_1 \pm \cdots \pm \zeta_{s-1} < 1.$$

Thus the asymptotic formula for $\mathcal{N}(P)$, proved for $s \ge 2^k + 1$, takes the form

$$\mathcal{N}(P) = \frac{C'_{k,s}}{(c_1 \cdots c_s)^{1/k}} P^{s-k} \mathfrak{S} + O(P^{s-k-\delta}), \tag{8.3}$$

where

$$C'_{k,s} = k^{-s} \int_0^1 \cdots \int_0^1 \{\eta_1 \cdots \eta_s(\eta_1 + \cdots - \eta_{s-1})\}^{-1+1/k}\, d\eta_1 \cdots d\eta_{s-1}, \tag{8.4}$$

and

$$\mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-s} S_{c_1 a,q} \cdots S_{-c_s a,q}. \tag{8.5}$$

We observe that the value of $\mathfrak{S}$ is now a number depending only on the coefficients $c_1, \ldots, -c_s$ and on k. As before, the series defining $\mathfrak{S}$ is absolutely convergent for $s \geq 2k+1$, and factorizes as $\prod \chi(p)$. Again there exists p_0 such that

$$\prod_{p>p_0} \chi(p) \geq \frac{1}{2}.$$

To ensure that $\mathfrak{S} > 0$ (there is now no need to write $\mathfrak{S} \geq C_1(k,s) > 0$, since there is no parameter N), it will suffice to prove that $\chi_p > 0$ for each individual p. As before, it suffices if

$$M(p^\nu) \geq C_p p^{\nu(s-1)} \tag{8.6}$$

for all sufficiently large ν, where $M(p^\nu)$ denotes the total number of solutions of the congruence

$$c_1x_1^k + \cdots - c_sx_s^k \equiv 0 \pmod{p^\nu}, \qquad 0 \leq x < p^\nu. \tag{8.7}$$

Our object now is to obtain some explicit function $s_1(k)$ of k, such that (8.6) holds for each p if $s \geq s_1(k)$. Then the asymptotic formula (8.3) will be significant, in the sense that the main term will be $\gg P^{s-k}$, and will imply that $\mathcal{N}(P) \to \infty$ as $P \to \infty$. In proving this result, the signs of the coefficients in (8.7) play no part, and therefore we revert to the original notation, in which there were no negative signs prefixed to the coefficients.

The first step is to derive a congruence in a smaller number of unknowns in which none of the coefficients are divisible by p, which is such that if (8.6) holds for the new congruence then it holds for the original congruence. We write

$$c_j = d_j p^{h_j k + l_j}, \quad (1 \leq j \leq s),$$

where

$$p \nmid d_j, \quad 0 \leq l_j < k.$$

Then (8.7) becomes

$$\sum_{j=1}^{s} d_j p^{l_j} \left(p^{h_j} x_j\right)^k \equiv 0 \pmod{p^\nu}.$$

Let $h = \max h_j$. We restrict ourselves to solutions of the form

$$x_j = p^{h-h_j} y_j.$$

Thus, for large ν, we can cancel p^{hk} from the congruence, and it becomes

$$\sum_{j=1}^{s} d_j p^{l_j} y_j^k \equiv 0 \pmod{p^{\nu-hk}}, \tag{8.8}$$

subject to

$$0 \le y < p^{\nu - h + h_j}.$$

If we denote by $M'(p^{\nu - hk})$ the number of solutions of (8.8) subject to

$$0 \le y < p^{\nu - hk}$$

then (since $h - h_j < hk$) we have

$$M(p^\nu) \ge M'(p^{\nu - hk}).$$

Hence it suffices to prove the analogue of (8.6) for $M'(p^\nu)$.

Let $l = \max l_j$. In the new congruence (8.8), but to the modulus p^ν, we group together the terms according to the value of l_j. There are k groups, and one at least of these must contain v terms, where $v \ge s/k$. We put $y_j = py'_j$ in the other terms, and after dividing out a factor p^l we obtain a congruence of the form

$$d_1 y_1^k + \cdots + d_v y_v^k + p(d_{v+1} y_{v+1}^k + \cdots) + \cdots \equiv 0 \pmod{p^{\nu - l}}. \quad (8.9)$$

Again we can replace $\nu - l$ on the right by ν, since this merely changes C_p in a result of the type (8.6). In the last congruence, we have

$$d_1 d_2 \cdots d_v \not\equiv 0 \pmod{p}.$$

Define γ as usual (see (5.11)). By the argument used in the proof of Lemma 5.5, the desired result (8.6) will hold for the congruence (8.9), provided the congruence

$$d_1 y_1^k + \cdots + d_v y_v^k \equiv 0 \pmod{p^\gamma} \quad (8.10)$$

has a solution in which $y_1, \ldots, y_v$ are not all divisible by p.

Suppose $p > 2$. We argue as in the proof of Theorem 7.2, dividing the terms in (8.10) into groups according to the class of the kth power residues or non-residues to which the coefficient d_j belongs. It suffices if

$$\frac{v}{p^{\gamma - 1}\delta} > 2k - 1,$$

and since $p^{\gamma - 1}\delta = p^\tau \delta \le k$, it suffices if

$$v > k(2k - 1).$$

Hence it suffices if

$$s > k^2(2k - 1).$$

Suppose $p = 2$. Once again there is no problem if $\tau = 0$, so that k is odd. If $\tau > 0$, we could argue as in the proof of Theorem 7.2, but there is a more effective argument which is quite simple. We shall prove that

$$d_1y_1^k + \cdots + d_vy_v^k \equiv 0 \pmod{2^\gamma}$$

has a solution with $y_1, \ldots, y_v$ not all even provided $v \geq 2^\gamma$. We find these solutions by taking $y_j = 0$ or 1 (this being no loss of generality when k is a power of 2, as remarked in connection with Lemma 5.6).

First, if $\gamma = 1$, we can solve

$$d_1t_1 + d_2t_2 \equiv 0 \pmod 2$$

by taking $t_1 = t_2 = 1$ (since d_1, d_2 are odd). Next, we can solve

$$d_1t_1 + d_2t_2 + d_3t_3 + d_4t_4 \equiv 0 \pmod 4$$

by taking either $t_1 = t_2 = 1$, $t_3 = t_4 = 0$ (if $d_1 + d_2 \equiv 0 \pmod 4$) or $t_1 = t_2 = 0$, $t_3 = t_4 = 1$ (if $d_3 + d_4 \equiv 0 \pmod 4$) or $t_1 = t_2 = t_3 = t_4 = 1$. The process continues, and the proof is easily completed by induction on γ.

The condition $v \geq 2^\gamma$ is satisfied if $v \geq 4k$, and therefore is satisfied if $s > k(4k-1)$. Since this number is less than $k^2(2k-1)$, it has no effect in the result.

Collecting our results, we have proved:

Theorem 8.1. *Let $c_1, \ldots, c_s$ be given integers, none of them 0, and not all of the same sign if k is even. Then provided $s \geq 2^k + 1$, and*

$$s \geq k^2(2k-1) + 1, \tag{8.11}$$

the equation

$$c_1x_1^k + \cdots + c_sx_s^k = 0$$

has infinitely many solutions in integers $x_1, \ldots x_s$, none of them 0.

The condition in (8.11), which came from our investigation of the singular series, is not best possible. In [27] Davenport and Lewis show that to ensure $\mathfrak{S} > 0$, it suffices if

$$s \geq k^2 + 1.$$

This condition is best possible if $k+1$ is a prime p. For then $x^k \equiv 1 \pmod p$ if $x \not\equiv 0 \pmod p$, and it is easily deduced that the congruence

$$\begin{aligned} \mathrm{p}^k \,|\, \big((x_1^k + \cdots + x_k^k) + p(x_{k+1}^k + \cdots + x_{2k}^k) + \cdots + p^{k-1} \\ \times (x_{k^2-k+1}^k + \cdots + x_{k^2}^k)\big)\,, \end{aligned}$$

in k^2 variables, is insoluble unless all the variables are divisible by p. However, for most values of k a smaller value than k^2+1 will suffice.

In the preceding treatment of the equation

$$c_1x_1^k + \cdots + c_sx_s^k = 0$$

we have obtained an asymptotic formula for the number of integer solutions in the s-dimensional box $0 < x_j \leq P_j$ as $P \to \infty$. But this box is not related in any unique way to the equation, and the interest of the result lies mainly in the fact that it establishes the existence of an infinity of solutions. To prove this, however, it is not essential to obtain an asymptotic formula for *all* solutions in such a box; it would be enough to consider some special subset. Thus we can use methods similar to those developed for the estimation of $G(k)$ in Waring's problem. In Chapter 9 we shall study Vinogradov's method, which is very effective for large k, and in the subsequent chapter we shall adapt this method to the equation which we have been studying.

It should not be overlooked, however, that the method which we have been using is particularly appropriate to the study of the *distribution* of the solutions of the equation. Suppose $\lambda_1, \ldots, \lambda_s$ are any real numbers, none of them 0, which satisfy the equation

$$c_1\lambda_1^k + \cdots + c_s\lambda_s^k = 0.$$

Then the method of the present section enables one to find an asymptotic formula, as $P \to \infty$, for the integral solutions of our equation in the box

$$1 - \delta < \frac{x_j}{\lambda_j P} < 1 + \delta,$$

for any small fixed positive number δ. Expressed geometrically, the result means that the 'rays' from the origin to the integer points on the cone

$$c_1x_1^k + \cdots + c_sx_s^k = 0$$

are everywhere dense on this cone, if the cone is considered as a real locus in s-dimensional space. Thus, although the method which is now to be expounded is more effective in establishing an infinity of solutions, it does not entirely supersede the previous method.