10

The equation $c_1x_1^k + \cdots + c_sx_s^k = 0$ again

We return to this equation, and adapt to it the method of the last section, so as to establish its solubility under a less restrictive condition than that of Theorem 8.1. As in Chapter 8 we shall suppose that $c_1, \ldots, c_s$ are fixed integers, none of them 0, and not all of the same sign if k is even.

Let $s_0 = s_0(k)$ be an integer which has the property that the singular series for any equation

$$c_1x_1^k + \cdots + c_{s_0}x_{s_0}^k = 0$$

is positive. By the work of Chapter 8, we can take

$$s_0 = k^2(2k-1) + 1, \tag{10.1}$$

and as remarked there it is in fact possible to take $s_0 = k^2 + 1$, though we have not proved this.

As in the preceding chapter, it will be necessary to have some knowledge about the singular series $\mathfrak{S}(M)$ of the related equation

$$c_1x_1^k + \cdots + c_{s_0}x_{s_0}^k = M. \tag{10.2}$$

This factorizes as $\prod_p \chi(p)$, and we know that

$$\prod_{p>p_0} \chi(p) \geq \frac{1}{2}$$

for some p_0 depending on k and s but not on M; this holds if s_0 is merely $\geq 2k+1$. Thus to ensure that $\mathfrak{S}(M)$ has a positive lower bound, independent of M, for some class of integers M, it will suffice if $\chi(p)$ has such a lower bound, for each $p \leq p_0$.

In the work of Chapter 8, we applied (for each p) a preliminary transformation to the additive form, depending on the powers of p dividing

the coefficients c_j. After this we obtained a form

$$d_1 y_1^k + \cdots + d_v y_v^k$$

with coefficients not divisible by p, and it was sufficient if the congruence

$$d_1 y_1^k + \cdots + d_v y_v^k \equiv 0 \pmod{p^\gamma}$$

had a solution with $y_1, \ldots, y_v$ not all divisible by p. We showed that this condition was satisfied if $s \geq s_0$, with s_0 in (10.1).

It follows that for each p there is some $\gamma_1(p)$ (depending also on the coefficients c_j) such that $\chi(p)$ has a positive lower bound if[1]

$$M \equiv 0 \pmod{p^{\gamma_1(p)}},$$

for this will ensure that the final congruence which has to be solved is the same as it would be for $M = 0$. Let

$$L = \prod_{p \leq p_0} p^{\gamma_1(p)}.$$

Then if $M \equiv 0 \pmod{L}$ there is a positive lower bound for $\chi(p)$, and hence also for $\mathfrak{S}(M)$, independent of the particular M.

Returning to the equation of the title, we take $s = s_0 + 3\ell$, with ℓ as in Chapter 9, and divide the coefficients into sets:

$$c_1, \ldots, c_{s_0}; \quad d_1, \ldots, d_\ell; \quad d'_1, \ldots, d'_\ell; \quad e_1, \ldots, e_\ell;$$

subject to the condition that $c_1, \ldots, c_{s_0}$ are not all of the same sign. We shall establish the solubility of the equation

$$c_1 x_1^k + \cdots + c_{s_0} x_{s_0}^k + L^k(u_1 + u_2 + y^k v) = 0, \tag{10.3}$$

subject to

(i) $0 < x_j \leq P/|c_j|^{1/k}$;
(ii) u_1 is an integer $< \frac{1}{4}(P/L)^k$, which is representable as $d_1 z_1^k + \cdots + d_\ell z_\ell^k$, and similarly for u_2 with accented coefficients;
(iii) $1 < y < (P/L)^{1/2k}$;
(iv) v is an integer $< \frac{1}{4}(P/L)^{k-1/2}$ which is representable as $e_1 t_1^k + \cdots + e_\ell t_\ell^k$.

This will prove the solubility of the original equation when $s = s_0 + 3\ell$, and *a fortiori* when $s \geq s_0 + 3\ell$.

The definition of major and minor arcs is the same as in Chapter 9, except that we replace $2kqP^{k-1}$ by $2kqcP^{k-1}$, where $c = \max|c_j|$; this

[1] We can take $\gamma_1(p) = \gamma$ if p does not divide any of the c_j.

is to ensure that the conditions of van der Corput's lemma are satisfied in the proof of the analogue of Lemma 9.2. In place of Lemma 9.3 we obtain that

$$\int_{\mathfrak{M}} T_1(\alpha)\cdots T_{s_0}(\alpha)e(-M\alpha)d\alpha \gg P^{s_0-k} \tag{10.4}$$

provided $0 < M \leq P^k$ and $M \equiv 0 \pmod{L}$. The proof of this differs only in respect of the singular integral, which transforms into a multiple of

$$\int_0^1 \cdots \int_0^1 \{\zeta_1 \cdots \zeta_{s_0-1}(\pm\zeta_1 \pm \cdots \pm \zeta_{s_0-1} - \theta)\}^{1-1/k} d\zeta_1 \cdots d\zeta_{s_0-1},$$

integrated over the range $0 < \pm\zeta_1 \pm \cdots \pm \zeta_{s_0-1} - \theta < 1$, where the signs are those of $c_1, \ldots, c_{s_0-1}$ and we have assumed that c_{s_0} is negative. We can suppose that two at least of the signs $\pm$ are $+$. Then, for $0 \leq \theta \leq 1$, the region of integration contains some small cube of size independent of θ, and it follows that the above integral has a positive lower bound.

We have already seen that the singular series occurring in the analogue of Lemma 9.3 has a positive lower bound for $M \equiv 0 \pmod{L}$, and hence we obtain (10.4).

Lemma 9.4 still applies, with slight changes, to give lower bounds for the number of integers of the form u or v. We consider the numbers u representable as

$$d_1z_1^k + z,$$

where z is representable as $d_2z_2^k + \cdots + d_\ell z_\ell^k$, and, if the variables z_1 and z are restricted to suitable ranges, the numbers are all distinct. The ranges depend of course on $d_1, \ldots, d_\ell$. We get the same lower bound for $U_\ell(X)$ as before, apart from a constant depending on $d_1, \ldots, d_\ell$.

We now need two exponential sums $R_1(\alpha)$, $R_2(\alpha)$ corresponding to the two different sets of numbers u_1, u_2; but since

$$\int_0^1 |R_1(\alpha)R_2(\alpha)|d\alpha \leq \left\{\int_0^1 |R_1(\alpha)|^2 d\alpha \int_0^1 |R_2(\alpha)|^2 d\alpha\right\}^{1/2},$$

we get the same saving as in the Corollary to Lemma 9.4.

The Corollary to Lemma 9.5 is essentially unchanged, and the proof is completed as before. Thus we obtain the following result.

Theorem 10.1. *Let $c_1, \ldots, c_s$ be integers, none of them 0 and not all of the same sign if k is even. Then, if*

$$s \geq s_0 + 3(2k\log 3k + 1), \tag{10.5}$$

where s_0 has the value given in (10.1), the equation

$$c_1 x_1^k + \cdots + c_s x_s^k = 0$$

has infinitely many solutions in integers $x_1, \ldots, x_s$, none of them 0.