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Cubic forms: minor arcs and major arcs

From now onwards we can suppose that alternative B of Lemma 13.4 holds. Thus, for any α, either

$$|S(\alpha)| < P^{n-\frac{1}{4}n\theta+\varepsilon} \tag{15.1}$$

or α lies in the set, which we shall call $\xi(\theta)$, of real numbers which have a rational approximation a/q satisfying

$$(a,q) = 1, \quad 1 \leq q \ll P^{2\theta}, \quad |q\alpha - a| \ll P^{-3+2\theta}. \tag{15.2}$$

We shall find that, provided $n \geq 17$, this result enables us to estimate satisfactorily the contribution made to the integral for $\mathcal{N}(P)$by all α outside $\xi(\theta_0)$, where θ_0 can be taken to be any fixed positive number, independent of P. Obviously it pays to take θ_0 small.

With this in mind, we define the *major arcs* $\mathfrak{M}$ to consist of the set $\xi(\theta_0)$, that is, the set of intervals (15.2) with θ replaced by θ_0, and we define the minor arcs $\mathfrak{m}$ to consist of the complement of this set with respect to the interval $0 < \alpha < 1$.

Lemma 15.1. *Provided $n \geq 17$, we have*

$$\int_{\mathfrak{m}} |S(\alpha)|\, d\alpha \ll P^{n-3-\delta} \quad (\delta > 0).$$

Proof. We choose a set of numbers

$$\theta_0 < \theta_1 < \cdots < \theta_h = \tfrac{3}{4} + \delta.$$

Every real α lies in the set $\xi(\theta_h)$, because we can always find a, q such that

$$q \leq P^{3/2}, \quad |q\alpha - a| < P^{-3/2},$$

and this implies that α is in $\xi(\frac{3}{4}+\delta)$. The minor arcs consist of the complement of $\xi(\theta_0)$, and we can regard this as the union of

$$\xi(\theta_h)-\xi(\theta_{h-1}),\quad \xi(\theta_{h-1})-\xi(\theta_{h-2}),\quad \cdots,\quad \xi(\theta_1)-\xi(\theta_0),$$

where the difference is meant in the sense of set theory.

In the set $\xi(\theta_g)-\xi(\theta_{g-1})$, the inequality (15.1) applies with $\theta=\theta_{g-1}$, so that

$$|S(\alpha)| \ll P^{n-\frac{1}{4}n\theta_{g-1}+\varepsilon}.$$

The measure of this set does not exceed the measure of $\xi(\theta_g)$, which is

$$\begin{aligned}&\ll \sum_{q\le P^{2\theta_g}} \sum_{a=1}^{q} q^{-1}P^{-3+2\theta_g}\\ &\ll P^{-3+4\theta_g}.\end{aligned}$$

Hence the contribution of this set to the integral of $|S(\alpha)|$ is

$$\ll P^{n-\frac{1}{4}n\theta_{g-1}-3+4\theta_g+\varepsilon}.$$

This is $\ll P^{n-3-\delta}$, since $n\ge 17$, provided

$$\theta_{g-1} > \frac{16}{17}\theta_g + \frac{4}{17}(\delta+\varepsilon).$$

We can choose $\theta_1,\ldots,\theta_h$ so near together that this is the case. Hence the result. □

Now we have to deal with the major arcs $\mathfrak{M}_{a,q}$, given by (15.2) with $\theta=\theta_0$. We put $2\theta_0=\Delta$, so that (15.2) becomes

$$(a,q)=1,\quad 1\le q\ll P^{\Delta},\quad |q\alpha-a|\le P^{-3+\Delta}.$$

It will be convenient to enlarge the major arcs slightly; we replace them by the intervals $\mathfrak{M}'_{a,q}$ in which the last inequality is divided by q on the left but not on the right:

$$(a,q)=1,\quad 1\le q\ll P^{\Delta},\quad |\alpha-a/q|\le P^{-3+\Delta}. \tag{15.3}$$

This is plainly permissible, for the contribution made by the additional set is a part of the contribution made by the minor arcs to $\int |S(\alpha)|\,d\alpha$, and is therefore covered by the estimate of Lemma 15.1.

The object of this enlargement is to make the length of the intervals independent of q, as in Chapter 4. It is only possible to do this when q is bounded by a small power of P, as here; but when it is possible, it leads to a slight simplification, in that the separation of the singular series from the singular integral takes place at an earlier point in the proof than it would do otherwise.

We define

$$S_{a,q} = \sum_{\mathbf{z} \pmod q} e\left(\frac{a}{q}C(\mathbf{z})\right),$$

$$I(\beta) = \int_{P\mathfrak{B}} e(\beta C(\xi))\ d\xi,$$

where $P\mathfrak{B}$ is the box used in the earlier definition of $S(\alpha)$ in Chapter 13.

Lemma 15.2. *For α in an interval $\mathfrak{M}'_{a,q}$ we have, on putting $\alpha = \beta + a/q$,*

$$S(\alpha) = q^{-n} S_{a,q} I(\beta) + O(P^{n-1+2\Delta}).$$

Proof. The crude argument used in the proof of Lemma 4.2 suffices, the reason being that here (as there) q is very small compared with P. We must now work in n dimensions instead of in one dimension, however, and this means replacing a sum by an integral. We must make an allowance for the discrepancy between the number of integer points in a large box and the volume of the box.

Putting $\mathbf{x} = q\mathbf{y} + \mathbf{z}$, where $0 \le z_j < q$, we have to estimate the difference between

$$\sum_{\mathbf{y}} e(\beta C(q\mathbf{y} + \mathbf{z})) \quad \text{and} \quad \int e(\beta C(q\boldsymbol{\eta} + \mathbf{z}))\ d\boldsymbol{\eta}$$

where the conditions of summation on $\mathbf{y}$ are such as will make $0 < qy_j + z_j < P$, and similarly for $\boldsymbol{\eta}$. Thus the edges of the box for $\mathbf{y}$ are $\ll P/q$, and the allowance mentioned above is $(P/q)^{n-1}$. We have also to allow for the variation of the integrand in a box of edge-length 1. We have

$$\left|\frac{\partial}{\partial \eta_j}\beta C(q\boldsymbol{\eta} + \mathbf{z})\right| \ll |\beta| q P^2 \ll q P^{-1+\Delta}.$$

The resulting error is obtained by multiplying by the volume of the region of integration, which is $\ll (P/q)^n$.

It follows that the difference between the sum and the integral is

$$\ll (P/q)^{n-1} + qP^{-1+\Delta} P^n q^{-n} \ll P^{n-1+\Delta} q^{1-n}.$$

(If n were equal to 1, this would be P^Δ, corresponding to P^δ in the proof of Lemma 4.2.) Summing over the q^n values of $\mathbf{z}$, the final error term is

$$\ll P^{n-1+\Delta}q \ll P^{n-1+2\Delta}.$$

$\square$

Lemma 15.3. *For a fixed cubic form $C(\mathbf{x})$ which does not represent zero, we have*

$$|S_{a,q}| \ll q^{\frac{7}{8}n+\varepsilon}.$$

Proof. We can apply alternative B of Lemma 13.4 to the sum $S(\alpha)$ with $\alpha = a/q$ and with $P = q$, and the box $\mathfrak{B}$ as the box $0 \le x_j < 1$, so that the box of summation becomes $0 \le x_j < q$. Take $\theta = \frac{1}{2} - \varepsilon$. Then either

$$|S(\alpha)| < q^{n-\frac{1}{4}n\theta+\varepsilon} \ll q^{\frac{7}{8}n+\varepsilon},$$

or α has a rational approximation a'/q' satisfying

$$1 \le q' \ll P^{1-2\varepsilon}, \quad |q'\alpha - a'| < P^{-2}.$$

But the latter is impossible when $\alpha = a/q$, for it would give then $q' < q$ and

$$|q'(a/q) - a'| < q^{-2},$$

whereas it is obvious that $|q'(a/q) - a'| \ge 1/q$. Hence the estimate for $S(\alpha) = S_{a,q}$ holds. $\square$

Lemma 15.4. *If $\mathfrak{M}'$ denotes the totality of the enlarged major arcs $\mathfrak{M}'_{a,q}$ then*

$$\int_{\mathfrak{M}'} S(\alpha)\, d\alpha = P^{n-3}\mathfrak{S}(P^\Delta)J(P^\Delta) + O(P^{n-3-\delta})$$

for some $\delta > 0$, where

$$\mathfrak{S}(P^\Delta) = \sum_{q \ll P^\Delta} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-n}S_{a,q},$$

$$J(P^\Delta) = \int_{|\gamma|<P^\Delta} \left(\int_{\mathfrak{B}} e(\gamma C(\boldsymbol{\xi}))\, d\boldsymbol{\xi} \right) d\gamma.$$

Proof. The error term in Lemma 15.2, when integrated over $|\beta| < P^{-3+\Delta}$ and summed over a and then over q gives a final error term

$$\ll \sum_{q \ll P^{\Delta}} \sum_{a} P^{-3+\Delta} P^{n-1+2\Delta} \ll P^{n-4+5\Delta}.$$

This is $\ll P^{n-3-\delta}$ provided Δ is sufficiently small.

The main term in Lemma 15.2 gives

$$\sum_{q \ll P^{\Delta}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-n} S_{a,q} \int_{|\beta|<P^{-3+\Delta}} I(\beta)\, d\beta.$$

The summation and integration are independent, and the summation gives $\mathfrak{S}(P^{\Delta})$. The integral becomes

$$P^{-3} \int_{|\gamma|<P^{\Delta}} I(P^{-3}\gamma)\, d\gamma,$$

and this is $P^{n-3} J(P^{\Delta})$, since

$$\begin{aligned} I(P^{-3}\gamma) &= \int_{P\mathfrak{B}} e\left(P^{-3}\gamma C(\boldsymbol{\xi})\right)\, d\boldsymbol{\xi} \\ &= P^{n} \int_{\mathfrak{B}} e\left(\gamma C(\boldsymbol{\xi})\right)\, d\boldsymbol{\xi}. \end{aligned}$$

Hence the result. $\square$