16

Cubic forms: the singular integral

The singular integral is the integral occurring in Lemma 15.4, namely

$$J(\mu) = \int_{-\mu}^{\mu} \left(\int_{\mathcal{B}} e(\gamma C(\boldsymbol{\xi})) d\boldsymbol{\xi} \right) d\gamma, \tag{16.1}$$

where we have written μ for P^{Δ}. It depends upon the box $\mathcal{B}$ which was used in the definition of the exponential sum $S(\alpha)$ in Chapter 13, and our object in the present chapter is to prove that we can choose $\mathcal{B}$ in such a way as to ensure that

$$J(\mu) \to J_0 > 0, \quad \text{as } \mu \to \infty. \tag{16.2}$$

We shall choose the box $\mathcal{B}$ so that it has for its centre a real solution $\xi_1^*, \ldots, \xi_n^*$ of the equation

$$C(\xi_1^*, \ldots, \xi_n^*) = 0. \tag{16.3}$$

This is a natural way to proceed, for our object is to obtain an asymptotic formula for $\mathcal{N}(P)$ which will show that $\mathcal{N}(P) \to \infty$; assuming that alternative A is excluded. If there were no real solution of $C(\boldsymbol{\xi}) = 0$ in $\mathcal{B}$, there would be none in $P\mathcal{B}$, and *a fortiori* $\mathcal{N}(P)$ would be 0.

We shall, in fact, take $\xi_1^*, \ldots, \xi_n^*$ to be rather more than an arbitrary real solution of $C(\boldsymbol{\xi}) = 0$; we shall take it to be a non-singular solution[1], in which none of $\xi_1^*, \ldots, \xi_n^*$ is 0. This is a convenient choice to ensure the truth of (16.2), and may even be essential.

The existence of such a solution is easily proved. For any real $\xi_2, \ldots, \xi_n$ we can find a real ξ_1 to satisfy $C(\xi_1, \ldots, \xi_n) = 0$, and it suffices to ensure that

$$\xi_1 \neq 0 \quad \text{and} \quad \partial C / \partial \xi_1 \neq 0.$$

[1] That is, one for which the partial derivatives of C are not all 0.

For any $\xi_2, \ldots, \xi_n$, the equation for ξ_1 is of the form

$$c_{111}\xi_1^3 + F\xi_1^2 + G\xi_1 + H = 0,$$

where F, G, H are forms in $\xi_2, \ldots, \xi_n$ of degrees $1, 2, 3$ respectively. Provided $H \neq 0$ (and we note that H cannot vanish identically) we shall have $\xi_1 \neq 0$. Let $D(\xi_2, \ldots, \xi_n)$ be the discriminant of the cubic equation in ξ_1. Then, provided $D \neq 0$, we shall have $\partial C/\partial \xi_1 \neq 0$. We can suppose that D does not vanish identically, for then the double root for ξ_1 is determined rationally by $\xi_2, \ldots, \xi_n$ and we get rational solutions of $C(\boldsymbol{\xi}) = 0$.

Thus we can find the desired non-singular real solution $\xi_1^*, \ldots, \xi_n^*$ of (16.3). We take $\mathcal{B}$ to be a small cube around this point, say

$$|\xi_j - \xi_j^*| < \rho, \quad (1 \leq j \leq n). \tag{16.4}$$

Lemma 16.1. *If ρ is chosen sufficiently small, (16.2) holds.*

Proof. We have

$$\begin{aligned} J(\mu) &= \int_{-\mu}^{\mu} \left(\int_{\mathcal{B}} e(\gamma C(\boldsymbol{\xi}) d\boldsymbol{\xi} \right) d\gamma \\ &= \int_{\mathcal{B}} \frac{\sin 2\pi\mu C(\boldsymbol{\xi})}{\pi C(\boldsymbol{\xi})} d\boldsymbol{\xi} \\ &= \int_{-\rho}^{\rho} \cdots \int_{-\rho}^{\rho} \frac{\sin 2\pi\mu C(\boldsymbol{\xi}^* + \boldsymbol{\eta})}{\pi C(\boldsymbol{\xi}^* + \boldsymbol{\eta})} d\boldsymbol{\eta}, \end{aligned} \tag{16.5}$$

where $\boldsymbol{\xi} = \boldsymbol{\xi}^* + \boldsymbol{\eta}$.

For any $\boldsymbol{\eta}$, we have

$$C(\boldsymbol{\xi}^* + \boldsymbol{\eta}) = c_1\eta_1 + \cdots + c_n\eta_n + P_2(\boldsymbol{\eta}) + P_3(\boldsymbol{\eta}), \tag{16.6}$$

where $P_2(\boldsymbol{\eta})$, $P_3(\boldsymbol{\eta})$ are forms of degrees $2, 3$ in $\boldsymbol{\eta}$. We have

$$c_1 = \frac{\partial C}{\partial \xi_1}(\xi_1^*, \ldots, \xi_n^*) \neq 0.$$

Without loss of generality we can suppose $c_1 = 1$.

For $|\boldsymbol{\eta}| < \rho$, we have

$$|C(\boldsymbol{\xi}^* + \boldsymbol{\eta})| < \sigma,$$

where $\sigma = \sigma(\rho)$ is small with ρ. Put $C(\boldsymbol{\xi}^* + \boldsymbol{\eta}) = \zeta$. Then, if ρ is sufficiently small, we can invert the relation (16.6) and express η_1 in terms of $\eta_2, \ldots, \eta_n$ by means of a power series. This will be one of the form

$$\eta_1 = \zeta - c_2\eta_2 - \cdots - c_n\eta_n + P(\zeta, \eta_2, \ldots, \eta_n),$$

where P is a multiple power series beginning with terms of degree 2 at least. Hence

$$\frac{\partial \eta_1}{\partial \zeta} = 1 + P_1(\zeta, \eta_2, \ldots, \eta_n),$$

and by taking ρ sufficiently small we can ensure that $|P_1| < 1/2$ for

$$|\eta_2| < \rho, \ldots, |\eta_n| < \rho, |\zeta| < \sigma.$$

Making a change of variable from η_1 to ζ in (16.5), we obtain

$$J(\mu) = \int_{-\sigma}^{\sigma} \frac{\sin 2\pi\mu\zeta}{\pi\zeta} V(\zeta) d\zeta, \tag{16.7}$$

where

$$V(\zeta) = \int_{\mathcal{B}'} \{1 + P_1(\zeta, \eta_2, \ldots, \eta_n)\}\, d\eta_2 \cdots d\eta_n,$$

in which $\mathcal{B}'$ denotes the part of the $(n-1)$-dimensional cube

$$|\eta_2| < \rho, \ldots, |\eta_n| < \rho$$

in which $|\eta_1| < \rho$, that is, in which

$$|\zeta - c_2\eta_2 - \cdots - c_n\eta_n + P(\zeta, \eta_2, \ldots, \eta_n)| < \rho.$$

It is clear that $V(\zeta)$ is a continuous function of ζ for $|\zeta|$ sufficiently small. It is also easily seen that $V(\zeta)$ is a function of bounded variation, since it has left and right derivatives at every value of ζ, and these are bounded. Hence, by Fourier's integral theorem applied to (16.7), we have

$$\lim_{\mu \to \infty} V(\mu) = V(0).$$

Now $V(0)$ is a positive number, for the cube $\mathcal{B}'$ contains any sufficiently small $(n-1)$-dimensional cube centred at the origin, and in such a cube we have $1 + P_1 > 1/2$. This proves the result. $\square$