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# Homogeneous equations of higher degree

In [6], Birch has given a far-reaching extension of the method by which we have treated the homogeneous cubic equation, but this involves some important modifications. He considers the problem of solving a homogeneous equation, or a system of simultaneous homogeneous equations (all of the same degree). Here one is faced by two serious difficulties. In the first place, even for a single equation of degree $k > 3$, we do not in general know of any reasonable function $n_0$ of $k$ which will be such that, if $n \geq n_0(k)$, the congruence conditions corresponding to the equation will be satisfied for every prime. (We know there is *some* function of $k$ for which the equation is soluble in the $p$-adic field, by the work of Brauer, quoted in Chapter 11, but this leads to an astronomical value. Moreover, a solution in the $p$-adic field would not be quite enough; we need a non-singular solution in order to be sure that we can satisfy the congruence conditions.) Thus we must postulate that the congruence conditions are satisfied for each prime $p$. We must also postulate that the equation, or system of equations, is soluble in the real number-field, with a non-singular solution.

Secondly — and this is more important — even these postulates are not always enough to ensure the solubility of the equation in integers (or rational numbers). The following example was shown to me by Swinnerton-Dyer;

$$3(x_1^2 + \cdots + x_r^2)^3 + 4(x_{r+1}^2 + \cdots + x_s^2)^3 = 5(x_{s+1}^2 + \cdots + x_n^2)^3 \quad (19.1)$$

where $r < s < n$. It is known from the work of Selmer [78] that the equation

$$3X^3 + 4Y^3 = 5Y^3$$

is insoluble except with $X = Y = Z = 0$, and it follows that (19.1) is insoluble except with $x_1 = \cdots = x_n = 0$. On the other hand it can be proved that (19.1) satisfies the congruence conditions for every $p$, and it is of course also soluble non-singularly in the real field.

Hence some further condition must be imposed if we are to establish solubility in integers. The type of condition which Birch is led to impose is expressed in terms of the dimension of a 'singular locus' associated with the system of equations.

We shall outline the general plan of his paper, giving comparisons with the problem of one cubic equation in places where this may be helpful. The details are somewhat formidable because of the inevitable complexity of the notation.

Suppose we have $R$ homogeneous forms of degree $k$ in $n$ variables, where $R < n$. We can write them as

$$\begin{aligned} f^{(1)}(\mathbf{x}) &= \sum_{j_0,\ldots,j_{k-1}} C^{(1)}_{j_0,\ldots,j_{k-1}} x_{j_0} \ldots x_{j_{k-1}}, \\ &\vdots \\ f^{(R)}(\mathbf{x}) &= \sum_{j_0,\ldots,j_{k-1}} C^{(R)}_{j_0,\ldots,j_{k-1}} x_{j_0} \ldots x_{j_{k-1}}, \end{aligned}$$

where the variables of summation go from 1 to $n$. Let $\mathfrak{B}$ be a box in $n$ dimensional space, and define the exponential sum

$$S(\alpha_1, \ldots, \alpha_R) = \sum_{\mathbf{x} \in P\mathfrak{B}} e\left(\alpha_1 f^{(1)}(\mathbf{x}) + \cdots + \alpha_R f^{(R)}(\mathbf{x})\right).$$

Then the number of integer points $\mathbf{x}$ in $P\mathfrak{B}$ which satisfy the simultaneous equations $f^{(1)}(\mathbf{x}) = 0, \ldots, f^{(R)}(\mathbf{x}) = 0$ is given by

$$\mathcal{N}(P) = \int_0^1 \cdots \int_0^1 S(\alpha_1, \ldots, \alpha_R)\, d\alpha_1 \ldots\, d\alpha_R.$$

By a straightforward extension of Lemma 13.1, we find that if

$$|S(\alpha_1, \ldots, \alpha_R)| \geq P^{n-K}, \quad (K > 0),$$

then

$$\begin{aligned} \sum_{\mathbf{x}^{(1)}} \cdots \sum_{\mathbf{x}^{(k-1)}} \prod_{J=1}^{n} \min\left\{P, \left\|\alpha_1 M_J^{(1)} + \cdots + \alpha_R M_J^{(R)}\right\|^{-1}\right\} \\ \gg P^{nk-2^{k-1}K}, \end{aligned}$$

where $M_J^{(1)}, \ldots, M_J^{(R)}$ are the multilinear forms in $k-1$ points $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k-1)}$ defined by

$$M_J^{(i)}\left(\mathbf{x}^{(1)} \mid \cdots \mid \mathbf{x}^{(k-1)}\right) = \sum_{j_1,\ldots,j_{k-1}} c^{(i)}_{J,j_1,\ldots,j_{k-1}} x^{(1)}_{j_1} \cdots x^{(k-1)}_{j_{k-1}},$$

for $i = 1, \ldots, R$. Lemma 13.1 itself is the case $R = 1$, $k = 3$. As in Lemma 13.2, it follows that the number of sets of $k-1$ integer points which satisfy

$$\left|\mathbf{x}^{(1)}\right| < P, \ldots, \left|\mathbf{x}^{(k-1)}\right| < P,$$
$$\left\|\alpha_1 M_J^{(1)} + \cdots + \alpha_R M_J^{(R)}\right\| < P^{-1}, \quad (1 \le J \le n),$$

is

$$\gg P^{(k-1)n-2^{k-1}K-\varepsilon}.$$

Using Lemma 12.6 $k-1$ times (instead of twice, as in the proof of Lemma 13.3), we deduce that the number of sets of $k-1$ integer points satisfying

$$\left|\mathbf{x}^{(1)}\right| < P^{\theta}, \ldots, \left|\mathbf{x}^{(k-1)}\right| < P^{\theta},$$
$$\left\|\alpha_1 M_J^{(1)} + \cdots + \alpha_R M_J^{(R)}\right\| < P^{-k+(k-1)\theta}$$

is

$$\gg P^{(k-1)n\theta-2^{k-1}K-\varepsilon}.$$

If there is any one of these sets of $k-1$ points for which the rank of the matrix

$$\begin{pmatrix} M_1^{(1)} & \cdots & M_1^{(R)} \\ \vdots & & \vdots \\ M_n^{(1)} & \cdots & M_n^{(R)} \end{pmatrix}$$

is $R$, then we get good rational approximations to $\alpha_1, \ldots, \alpha_R$, all with the same denominator $q$. This denominator arises as the value of some determinant (non-zero) of order $R$ in the above matrix. In fact we get

$$|q\alpha_i - a_i| \ll P^{-k+R(k-1)\theta}$$

and

$$q \ll P^{R(k-1)\theta}.$$

The exponents here correspond to $-3 + 2\theta$ and $2\theta$ respectively, as in alternative B of Lemma 13.4.

The real difficulty is when the above fails, i.e., when the rank of the above matrix is $\leq R-1$ for all sets $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k-1)}$. In the case $R=1$ this would mean that the multilinear forms $M_J$ all vanish at all these sets of integer points.

The main new idea of Birch's paper is to express this possibility in terms of dimensions of varieties. We regard a set of $k-1$ points as a single point in a space of $(k-1)n$ dimensions. The condition that the rank of the above matrix shall be $\leq R-1$ defines an algebraic variety in this space; and from the lower bound for the number of integer points on it, we deduce that the dimension of this variety is

$$\geq (k-1)n - 2^{k-1}K/\theta + \varepsilon.$$

It is a simple principle of algebraic geometry that the dimension of a variety (that is, the maximum dimension of any of its absolutely irreducible components) cannot be reduced by more than $t$ if we pass to the intersection of the variety with a linear space defined by $t$ equations. Hence the intersection of the above variety with the 'diagonal' linear space

$$\mathbf{x}^{(1)} = \mathbf{x}^{(2)} = \cdots = \mathbf{x}^{(k-1)},$$

defined by $(k-2)n$ equations, has dimension

$$\geq n - 2^{k-1}K/\theta - \varepsilon.$$

If $\mathbf{x} = \mathbf{x}^{(1)} = \cdots = \mathbf{x}^{(k-1)}$, the new variety consists of all points $\mathbf{x}$ for which the rank of the matrix

$$\begin{pmatrix} \frac{\partial f^{(1)}}{\partial x_1} & \cdots & \frac{\partial f^{(R)}}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f^{(1)}}{\partial x_n} & \cdots & \frac{\partial f^{(R)}}{\partial x_n} \end{pmatrix}$$

is $\leq R-1$. We call this *the singular locus associated with the given equations*, and denote it by $V^*$. Thus the present case leads to

$$\dim V^* \geq n - 2^{k-1}K/\theta - \varepsilon.$$

If $\dim V^* = s$, we can prevent this happening (and thereby exclude the situation which now corresponds to alternative A of Lemma 13.4) by choosing

$$K = \frac{\theta}{2^{k-1}}(n - s - 2\varepsilon).$$

Having made this choice, we have a situation similar to that of alternative B; that is, for each $\alpha_1, \ldots, \alpha_R$ there is either an estimate for $|S(\alpha_1, \ldots, \alpha_R)|$ or a good set of simultaneous approximations to $\alpha_1, \ldots, \alpha_R$. This proves the basis for a treatment similar in principle to that of Chapters 15, 16, 17 for one cubic equation.

The main difficulty lies with the singular integral, and here again the dimension of the singular variety comes in. The treatment of the integral is too elaborate to be outlined here. It is essential to suppose that the original equations define a variety of dimension $n - R$.

The result of Birch's paper is as follows:

**Theorem 19.1.** *Let $f_1, \ldots, f_R$ be forms of degree $k$ in $n$ variables with integral coefficients, where $n > R \geq 1$. Let $V$ denote the algebraic variety*

$$f_1(\mathbf{x}) = 0, \ldots, f_R(\mathbf{x}) = 0,$$

*and suppose $V$ has dimension $n - R$. Let $V^*$ be the associated singular locus, and let $s = \dim V^*$. Suppose there is a non-singular real point on $V$, and a non-singular $p$-adic point on $V$ for every prime $p$. Then provided*

$$n - s > R(R+1)(k-1)2^{k-1},$$

*there is an integer point $\mathbf{x} \neq \mathbf{0}$ on $V$.*