20

A Diophantine inequality

In the subject of Diophantine inequalities, our aim is to solve in integers some given type of inequality, and usually one involving polynomials or forms with arbitrary real coefficients. The geometry of numbers provides useful methods for investigating the solubility of linear inequalities, and gives some information about inequalities involving polynomials of higher degree, but is limited in its power in relation to the latter.

The simplest Diophantine inequality of higher degree than the first is

$$|\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2| < C.$$

On the basis of analogy with Meyer's theorem which was encountered in Chapter 11, it was conjectured by Oppenheim in 1929 that provided $n \geq 5$, the inequality should be soluble for *all* $C > 0$, provided that $\lambda_1, \ldots, \lambda_n$ are real numbers which are not all of the same sign. Of course, if $\lambda_1, \ldots, \lambda_n$ are in rational ratios, we can make the left-hand side zero, so the problem relates to the case in which the ratios are not all rational.

In 1934 the result was proved to hold if $n \geq 9$ by Chowla [13]; he deduced it from results of Jarník and Walfisz [51] on the number of integer points in a large ellipsoid. In 1945 it was proved to hold for $n \geq 5$ by Davenport and Heilbronn [26], and the present chapter is mainly devoted to an account of the proof. It should be noted that although the bound 5 for the number of variables is in some sense best possible, there is a deeper sense in which it is probably not. If we assume that the ratios λ_i/λ_j are not all rational, the result may (for all we know to the contrary) hold for $n \geq 3$.

Stated formally, the result to be proved is:

Theorem 20.1. *Let $\lambda_1, \ldots, \lambda_5$ be real numbers, none of them 0, and* *not all positive nor all negative. Suppose that one at least of the ratios* λ_i/λ_j *is irrational. Then, for any* $\varepsilon > 0$ *there exist integers* $x_1, \ldots, x_5$, *not all* 0, *such that*

$$|\lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2| < \varepsilon.$$

We can suppose without loss of generality that

$$\lambda_1 > 0, \quad \lambda_5 < 0, \quad \lambda_1/\lambda_2 \notin \mathbb{Q}.$$

It will suffice to prove the solubility of

$$|\lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2| < 1, \tag{20.1}$$

since then the solubility of the apparently more general inequality follows on replacing $\lambda_1, \ldots, \lambda_5$ in the last inequality by $\lambda_1/\varepsilon, \ldots, \lambda_5/\varepsilon$.

The first step is to construct a function of a real variable Q which is positive for $|Q| < 1$ and zero for $|Q| \geq 1$. One such is given in the following lemma, but there are various other similar ones.

Lemma 20.1. *We have*

$$\int_{-\infty}^{\infty} e(\alpha Q) \left(\frac{\sin \pi\alpha}{\pi\alpha} \right)^2 d\alpha = \begin{cases} 1 - |Q|, & |Q| \leq 1, \\ 0, & |Q| \geq 1. \end{cases}$$

Proof. It is well known that

$$\int_{-\infty}^{\infty} \left(\frac{\sin \pi\alpha}{\pi\alpha} \right)^2 d\alpha = 1.$$

Hence

$$\int_{-\infty}^{\infty} \left(\frac{\sin \pi\eta\alpha}{\pi\alpha} \right)^2 d\alpha = |\eta|$$

for any real η. This gives

$$\begin{aligned} &\int_{-\infty}^{\infty} e(\alpha Q) \left(\frac{\sin \pi\alpha}{\pi\alpha} \right)^2 d\alpha \\ &= \int_{-\infty}^{\infty} \cos 2\pi\alpha Q \left(\frac{\sin \pi\alpha}{\pi\alpha} \right)^2 d\alpha \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin^2 \pi\alpha(Q+1) + \sin^2 \pi\alpha(Q-1) - 2\sin^2 \pi\alpha Q}{(\pi\alpha)^2} d\alpha \\ &= \frac{1}{2} \left\{ |Q+1| + |Q-1| - 2|Q| \right\}, \end{aligned}$$

which gives the result. $\square$

Let P be a large positive integer. Define

$$S(\alpha) = \sum_{x=1}^{P} e(\alpha x^2), \quad I(\alpha) = \int_0^P e(\alpha x^2) dx.$$

Taking $Q = \lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2$ in the result of Lemma 20.1 and summing over $x_1, \ldots, x_5$ we get

$$\int_{-\infty}^{\infty} S(\lambda_1\alpha)\cdots S(\lambda_5\alpha)\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 d\alpha = \sum_{\substack{x_1,\,\ldots,\,x_5 \\ |Q|\,<\,1}} (1-|Q|), \quad (20.2)$$

where the summation is over integers with $1 \le x_j \le P$ subject to (20.1). Similarly, integrating over $x_1, \ldots, x_5$ instead of summing, we get

$$\int_{-\infty}^{\infty} I(\lambda_1\alpha)\cdots I(\lambda_5\alpha)\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 d\alpha = \int\cdots\int(1-|Q|)dx_1\cdots dx_5, \quad (20.3)$$

where the integration is over real variables with $0 \le x_j \le P$ subject to (20.1).

The general idea of the proof is to compare (20.2) with (20.3). It will be an easy matter to prove that the right-hand side of (20.3) is $\gg P^3$ as $P \to \infty$ (see Lemma 20.2 below). If we could prove that the left-hand sides of (20.2) and (20.3) differ by an amount which is $o(P^3)$ as $P \to \infty$, it would follow that the right-hand side of (20.2) was $\gg P^3$. This would imply that there are $\gg P^3$ integral solutions $(x_1, \ldots, x_5)$ of (20.1), with $1 \le x_j \le P$.

We shall prove that there is a small interval around $\alpha = 0$ in which $S_j(\alpha)$ differs very little from $I_j(\alpha)$, and from this we shall deduce that the contributions made by this interval to the two integrals are effectively the same (Lemma 20.4). It will be easy to prove that all other α make a negligible contribution to the integral on the left-hand side of (20.3). The difficulty lies in estimating the contribution made by such α to the integral on the left side of (20.2). Here (and here only) we use the hypothesis that λ_1/λ_2 is irrational, and we shall not prove the result in question for *all* large P but only for a particular sequence.

Lemma 20.2. *We have*

$$\int_{-\infty}^{\infty} I(\lambda_1\alpha)\cdots I(\lambda_5\alpha)\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 d\alpha \gg P^3.$$

Proof. In the right-hand side of (20.3) we put $|\lambda_i|x_i^2 = y_i$. The integral becomes (apart from the constant factor)

$$\int_0^{|\lambda_1|P^2} \cdots \int_0^{|\lambda_5|P^2} \{1 - |y_1 \pm y_2 \pm \cdots - y_5|\} (y_1 \cdots y_5)^{-1/2} dy_1 \cdots dy_5,$$

where the integral is over $y_1, \ldots, y_5$ for which $|y_1 \pm y_2 \pm \cdots - y_5| < 1$ and the signs are those of $\lambda_1, \ldots, \lambda_5$.

We limit the variables y_2, y_3, y_4 to the interval $\frac{1}{2}\gamma P^2 < y_j < \gamma P^2$, and we limit y_5 to the interval $4\gamma P^2 < y_5 < 5\gamma P^2$, and we limit y_1 to the interval

$$|y_1 \pm y_2 \pm y_3 \pm y_4 - y_5| < \frac{1}{2}.$$

Then all the remaining points have $0 < y_j < |\lambda_j|P^2$ provided $9\gamma < \min|\lambda_j|$. Hence we have a portion of the domain of integration, of volume $\gg (P^2)^4$. In this domain, the integrand is

$$\gg (y_1 \cdots y_5)^{-1/2} \gg (P^{10})^{-1/2}.$$

Hence the integral is $\gg P^3$. $\square$

Lemma 20.3. *If* $|\alpha| < (4\lambda P)^{-1}$, *where* $\lambda = \max|\lambda_j|$, *then*

$$S(\lambda_j\alpha) = I(\lambda_j\alpha) + O(1).$$

Proof. This is a case of Lemma 9.1 (van der Corput's lemma), with $f(x) = \lambda_j\alpha x^2$. We have

$$|f'(x)| = |2\lambda_j\alpha x| \le 2|\lambda_j\alpha|P \le \frac{1}{2},$$

and $f''(x)$ is of fixed sign. $\square$

Lemma 20.4. *We have*

$$\int_{|\alpha|<(4\lambda P)^{-1}} S(\lambda_1\alpha)\cdots S(\lambda_5\alpha)\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 d\alpha \gg P^3.$$

Proof. First we note that, for any α,

$$|I(\lambda_j\alpha)| \ll \min(P, |\alpha|^{-1/2}).$$

The estimate P is obvious, and the estimate $|\alpha|^{-1/2}$ follows from

$$I(\lambda_j\alpha) = \int_0^P e(\lambda_j\alpha x^2)dx = \frac{1}{2}|\lambda_j\alpha|^{-1/2}\int_0^{|\lambda_j\alpha|P^2} t^{-1/2}e(\pm t)dt,$$

since the last integral is bounded. It follows from Lemma 20.3 that if $|\alpha| < (4\lambda P)^{-1}$ then

$$|S(\lambda_j \alpha)| \ll \min(P, |\alpha|^{-1/2})$$

also. Hence, using these two estimates in conjunction with Lemma 20.3, we have

$$|S(\lambda_1 \alpha) \cdots S(\lambda_5 \alpha) - I(\lambda_1 \alpha) \cdots I(\lambda_5 \alpha)| \ll \min(P^4, \alpha^{-2})$$

in the above interval. Hence the integral of the difference over $|\alpha| < (4\lambda P)^{-1}$ is $O(P^2)$.

Hence it suffices to prove that

$$\int_{|\alpha|<(4\lambda P)^{-1}} I(\lambda_1 \alpha) \cdots I(\lambda_5 \alpha) \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha \gg P^3.$$

We already know that this is true for the corresponding integral over $(-\infty, \infty)$. Now by the above estimate for $I(\lambda_j \alpha)$, we have

$$\int_{|\alpha|\geq(4\lambda P)^{-1}} |I(\lambda_1 \alpha) \cdots I(\lambda_5 \alpha)| d\alpha \ll \int_{|\alpha|\geq(4\lambda P)^{-1}} \alpha^{-5/2} d\alpha \ll P^{3/2}.$$

Hence the result is proved. □

We now come to the heart of the problem; that is, the estimation of

$$\int_{|\alpha|\geq(4\lambda P)^{-1}} |S(\lambda_1 \alpha) \cdots S(\lambda_5 \alpha)| \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha.$$

Lemma 20.5. *For any $\varepsilon > 0$ we have*

$$\int_0^1 |S(\alpha)|^4 d\alpha \ll P^{2+\varepsilon}.$$

Proof. By the definition of $S(\alpha)$, the integral equals the number of solutions of $x_1^2 + x_2^2 = y_1^2 + y_2^2$ in integers between 1 and P inclusive. The number of solutions with $x_2 = y_2$ is P^2. In other solutions, the value of x_2 and y_2 determine those of $x_1 - y_1$ and $x_1 + y_1$ with $\ll P^\varepsilon$ possibilities, since these are factors of $x_2^2 - y_2^2$. Hence the result. □

Lemma 20.6. *For any fixed $\delta > 0$ we have*

$$\int_{|\alpha|\geq P^\delta} |S(\lambda_1 \alpha) \cdots S(\lambda_5 \alpha)| \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha \ll P^{3-\delta/2}.$$

Proof. In view of the trivial estimate $|S(\lambda_5\alpha)| \le P$, it suffices to prove that

$$\int_{|\alpha|\ge P^\delta} |S(\lambda_1\alpha)\cdots S(\lambda_4\alpha)| \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha \ll P^{2-\delta/2}.$$

By Hölder's inequality, it suffices to prove that

$$\int_{|\alpha|\ge P^\delta} |S(\lambda_j\alpha)|^4 \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha \ll P^{2-\delta/2},$$

and hence it suffices to prove that

$$\int_{|\alpha|\ge P^\delta} |S(\alpha)|^4 \frac{d\alpha}{\alpha^2} \ll P^{2-\delta/2}.$$

Since $S(\alpha)$ is periodic with period 1, Lemma 20.5 implies that

$$\int_m^{m+1} |S(\alpha)|^4 d\alpha \ll P^{2+\varepsilon}.$$

Hence, if $M = [P^\delta]$, the integral in question is at most

$$\begin{aligned} \sum_{m=M}^{\infty} \int_m^{m+1} |S(\alpha)|^4 \frac{d\alpha}{\alpha^2} &\ll P^{2+\varepsilon} \sum_{m=M}^{\infty} \int_m^{m+1} \frac{d\alpha}{\alpha^2} \\ &\ll P^{2+\varepsilon} M^{-1} \\ &\ll P^{2-\delta+\varepsilon} \\ &\ll P^{2-\delta/2}. \end{aligned}$$

Hence the result. □

By (20.2) and Lemma 20.4 and Lemma 20.6, it suffices to prove that

$$\int_{(4\lambda P)^{-1}<|\alpha|<P^\delta} |S(\lambda_1\alpha)\cdots S(\lambda_5\alpha)| d\alpha = o(P^3).$$

For then the right-hand side of (20.2) is $\gg P^3$, and this is what we want to prove. As mentioned earlier, we can only prove this for certain restricted values of P.

It follows from Lemma 20.5 and Hölder's inequality that

$$\int_{(4\lambda P)^{-1}<|\alpha|<P^\delta} |S(\lambda_{i_1}\alpha)\cdots S(\lambda_{i_4}\alpha)| d\alpha \ll P^{2+\delta+\varepsilon}$$

for any four different subscripts $i_1, \ldots, i_4$. Using this with the subscripts $2, 3, 4, 5$ and $1, 3, 4, 5$, we see that it will suffice if, for each α in the range of integration, we have

$$\min\left(|S(\lambda_1\alpha)|, |S(\lambda_2\alpha)|\right) \ll P^{1-2\delta}. \tag{20.4}$$

For this we must use the irrationality of λ_1/λ_2.

Lemma 20.7. *Suppose that* $(a, q) = 1$ *and* $|\alpha - a/q| < q^{-2}$. *Then*

$$|S(\alpha)| \ll P^{1+\varepsilon}\left\{P^{-1/2} + q^{-1/2} + \left(\frac{P^2}{q}\right)^{-1/2}\right\}.$$

Proof. This is Lemma 3.1 (Weyl's inequality), with $k = 2$, and consequently also $K = 2$. □

We choose any convergent a_0/q_0 to the continued fraction for λ_1/λ_2, and therefore have

$$\left|\frac{\lambda_1}{\lambda_2} - \frac{a_0}{q_0}\right| < \frac{1}{q_0^2}. \tag{20.5}$$

We take $P = q_0^2$; this limits P to an infinite sequence of values.

Lemma 20.8. *With P restricted as above, the estimate (20.4) holds for each α in the range* $(4\lambda P)^{-1} < \alpha < P^\delta$.

Proof. There exists a rational approximation a_1/q_1 to $\lambda_1\alpha$ such that

$$(a_1, q_1) = 1, \quad 1 \le q_1 \le P^{3/2}, \quad \left|\lambda_1\alpha - \frac{a_1}{q_1}\right| < \frac{1}{q_1 P^{3/2}}. \tag{20.6}$$

We observe that $a_1 \ne 0$, for if $a_1 = 0$ then $|\lambda_1\alpha| < P^{-3/2}$, contrary to hypothesis. Similarly there exists a rational approximation a_2/q_2 to $\lambda_2\alpha$ such that

$$(a_2, q_2) = 1, \quad 1 \le q_2 \le P^{3/2}, \quad \left|\lambda_2\alpha - \frac{a_2}{q_2}\right| < \frac{1}{q_2 P^{3/2}}, \tag{20.7}$$

and again $a_2 \ne 0$.

If $q_1 > P^{5\delta}$, we can apply Lemma 20.7 to $S(\lambda_1\alpha)$, with a_1, q_1 in place of a, q, and this gives[1]

$$|S(\lambda_1\alpha)| \ll P^{1+\varepsilon-5\delta/2} \ll P^{1-2\delta}.$$

Similarly if $q_2 > P^{5\delta}$ we get the analogous result for $|S(\lambda_2\alpha)|$. In either of these events, (20.4) is satisfied. So we can suppose that

$$q_1 \le P^{5\delta}, \quad q_2 \le P^{5\delta}. \tag{20.8}$$

[1] We assume that δ is small.

We can now deduce from (20.6) and (20.7) that λ_1/λ_2 is well approximated by $a_1 q_2/a_2 q_1$. We note that since a_1/q_1 is an approximation to $\lambda_1\alpha$, and $|\alpha| < P^{\delta}$, we have $|a_1| \ll P^{6\delta}$, and similarly $|a_2| \ll P^{6\delta}$. Hence

$$\frac{\lambda_1}{\lambda_2} = \frac{\lambda_1\alpha}{\lambda_2\alpha} = \frac{\frac{a_1}{q_1}\left(1 + O(P^{-3/2})\right)}{\frac{a_2}{q_2}\left(1 + O(P^{-3/2})\right)} = \frac{a_1 q_2}{a_2 q_1}\left(1 + O(P^{-3/2})\right),$$

and since $|a_1 q_2/a_2 q_1|$ is bounded above, this implies

$$\left|\frac{\lambda_1}{\lambda_2} - \frac{a_1 q_2}{a_2 q_1}\right| \ll P^{-3/2}.$$

We have $1 \le |a_2| q_1 \ll P^{11\delta}$.

Comparison of the last result with (20.5) gives a contradiction if δ is sufficiently small (and q_0 is sufficiently large). For we get

$$\left|\frac{\lambda_1}{\lambda_2} - \frac{a_1 q_2}{a_2 q_1}\right| \ll P^{-3/2} + q_0^{-2} \ll q_0^{-2},$$

for $P = q_0^2$, whereas the left-hand side is

$$\ge \frac{1}{q_0 |a_2| q_1} \gg \frac{1}{q_0 P^{11\delta}} \gg q_0^{-1-6\delta}.$$

This completes the proof of Lemma 20.8; and by our earlier remarks, Lemma 20.8 completes the proof of Theorem 20.1. □

Certain extensions of Theorem 20.1 are almost immediate. First, we could prove the same result for the inequality

$$|\lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2 - \mu| < \varepsilon,$$

for any real number μ (assuming that the ratios λ_i/λ_j are not all rational). Secondly we could replace the squares by kth powers, provided the number of variables is at least $2^k + 1$. In this case we should use Hua's inequality (Lemma 3.2) in place of the above Lemma 20.7. More precise results have been proved by Davenport and Roth [30] and by Danicic [16].

The extension to Diophantine inequalities involving general forms with real coefficients, in place of additive forms, present perhaps even more difficulty than the corresponding extension for Diophantine equations.

All the results so far obtained depend on results for Diophantine equations, and usually one needs these in a more precise form, in which there is an estimate for the size of a solution.

The first problem that naturally presents itself is that of establishing the solubility of

$$|Q(x_1, \ldots, x_n)| < \varepsilon \tag{20.9}$$

for any $\varepsilon > 0$, where Q is any indefinite quadratic form. By some very complicated work, this has been proved to hold if $n \geq 21$, the work being the result of joint efforts by Birch, Davenport and Ridout [70]. By a result of Oppenheim [64, 65], it follows that if Q is not proportional to a form with integral coefficients, then the inequality

$$|Q(x_1, \ldots, x_n) - \mu| < \varepsilon,$$

for any real μ, is soluble. Thus the values of any real indefinite quadratic form in 21 or more variables are either discrete (if the form is proportional to an integral form) or everywhere dense.

An analogue of (20.9) for any real cubic form has been proved by Pitman [66], but the number of variables needed is fairly large. There seems to be a difficulty of principle in proving any analogous result for a form of degree five.