V. THE LATER CONSCIOUS WORK

The Fourth Stage. We have now become acquainted with the three stages in invention which Helmholtz and Poincaré have taught us to distinguish: preparation, incubation and illumination. But Poincaré shows the necessity and importance of a fourth and final one, which again occurs in consciousness. This new intervention of consciousness, after the unconscious work, is necessary not only for the obvious purpose of expressing the results by language or writing, but at least for three other reasons which are, however, closely dependent on each other:

(1) To Verify Them. The feeling of absolute certitude which accompanies the inspiration generally corresponds to reality; but it may happen that it has deceived us.[1] Whether such has been the case or not must be ascertained by our properly so-called reason, a task which belongs to our conscious self.

(2) To "Precise" Them. That is, to state them precisely. It never happens, as Poincaré observes, that the unconscious work gives us the results of a somewhat long calculation already solved in its entirety. If we should, as concerns the unconscious, retain the original idea which is suggested by the quality "automatic" imputed to it, we should suppose that, thinking of an algebraic calculation before falling asleep, we might hope to find its result ready made upon our awakening; but nothing of the sort ever happens, and indeed we begin to understand that automa-

[1] Poincaré notices that this happens to him especially in regard to ideas coming to him in the morning or evening, or in bed while in a semihypnagogic state.

tism of the unconscious must not be understood in that way. On the contrary, effective calculations which require discipline, attention and volition, and, therefore, consciousness, depend on the second period of conscious work which follows the inspiration.

Thus, we come to the paradoxical-looking conclusion—to which, besides, we shall have to bring a correction as we already have done in Newton's case—that this intervention of our will, i.e., of one of the highest faculties of our soul, happens in a rather mechanical part of the work, where it is in some way subordinated to the unconscious, though supervising it. The second operation is inseparable from the first, from verification. The conscious mind performs them both at the same time.

A Statement by Paul Valéry. What we just met with in the domain of mathematical research, especially that coordination of the "precising" work to the original inspiration, is once more in agreement with what Paul Valéry says on quite a different kind of invention, except that the very description of Paul Valéry suggests that facts may be even more complicated or delicate than he himself or Poincaré saw them, and would deserve a more thorough study. Paul Valéry says, in the passage[2] of which we have already quoted the beginning (see Section I, p. 17):

"There is the period of dark-room. There must be no excessive zeal at this moment, or you would spoil your plate. You must have your reagents, you must work as your own employee, your own foreman. The master has provided the spark, it is your job to make something of it. A very curious thing is the disappointment that may follow. There are misleading gleams of light; when the foreman comes to the

[2] *Bulletin Soc. Philosophie,* Vol. 28 (1928), p. 16.

result, he perceives that there is no authentic product, that it would have been good *if* it had been true. Sometimes a series of judgments intervene which cancel each other out. A kind of irritation follows; you say yourself you will never succeed in recording what appears before you."

This "precising" state in invention is again quite a general one, and even the most spontaneous creators experience it. The same Lamartine whom we have seen to answer so rapidly, unhesitatingly and as though almost unwillingly when asked for a couple of verses, is reported by his biographers to have repeatedly and indefatigably corrected his work, as appears from his manuscripts.

Numerical Calculators. The process seems to be slightly different, on one point, in a case which one is often tempted to mix with that of mathematicians: I mean those prodigious calculators—frequently quite uneducated men—who can very rapidly make very complicated numerical calculations, such as multiplications of numbers in ten or more figures, who will want only one instant of reflection to tell you how many minutes or seconds have elapsed since the beginning of our era.

Such a talent is, in reality, distinct from mathematical ability. Very few known mathematicians are said to have possessed it: one knows the case of Gauss and Ampère and also in the seventeenth century, Wallis. Poincaré confesses that he is a rather poor numerical calculator, and so am I.

Exceptional calculators often present remarkable psychological peculiarities.[3] The one I want to mention as belonging to our subject is that, contrary to what we just heard from Poincaré, it happens that calculation results,

[3] We mention the very curious one that, in several of them, that ability has been temporary and disappeared after some years.

or at least partial ones, appear to them without willful effort and by inspiration elaborated in their unconscious.

Perhaps the most outspoken testimony is afforded by a letter written to Möbius[4] by the calculator Ferrol: "If I was asked any question, rather a difficult one by itself, the result immediately proceeded from my sensibility without my knowing at the first moment how I had obtained it; starting from the result, I then sought the way to be followed for this purpose. That intuitive conception which, curiously enough, has never been shaken by an error, has developed more and more as needs increased. Even now, I have often the sensation of somebody beside me whispering the right way to find the desired result; it concerns some ways where few people have entered before me and which I should certainly not have found if I had sought for them by myself.

"It often seems to me, especially when I am alone, that I find myself in another world. Ideas of numbers seem to live. Suddenly, questions of any kind rise before my eyes with their answers."

It must be added that Ferrol was attracted not only by numerical calculations, but also and even more strongly by algebraic ones. It is the more striking that, also in that case, he brings calculations to an effective end in an unconscious way.[5]

[4] See *Die Anlage für Mathematik*, pp. 74-76.

[5] Unconscious interventions in numerical calculations are also reported by Scripture, *American Journal of Psych.*, Vol. IV (1891). See also Binet, *Psychologie des Grands Calculateurs et Joueurs d'Echecs*. However, these statements are not as positive and precise as Ferrol's, and confusion would be possible between partial results unconsciously obtained and results known in advance by heart.

Appreciation of One's Own Work. Once we have obtained our result, what do we think of it?

Very often research which has deeply interested me while I was investigating it loses its interest for me just after I have the solution, unhappily at a time which coincides with the period when I have to record it. After a while, say a couple of months, I come to a more just appreciation of it.

Paul Valéry was asked the same question about his feeling toward his own work after its completion at a meeting of the Société de Philosophie in Paris; he answered: "It always turns out badly; *Je divorce*"; and he already gave an indication in the same sense when describing the invention process, as we have seen.

(3) The Continuation of the Work. Relay-Results. The double operation of verifying and "precising" the result assumes another meaning when, as happens most frequently, we regard it not as the end of the research, but as one stage of it—we have met with such successive stages in Poincaré's report—so that we think of *utilizing* it.

Such a utilization not only requires that the result be verified, but that it be "precised." Indeed, since we know that our unconscious work, showing us the way to obtain the result, does not offer it in its precise form, it may happen, and it actually happens in many cases, that some features in that precise form, which we could not fully foresee, wield a capital and even total influence on the continuation of the thought.

Such has already been the case with Poincaré's initial stage (though not for the following ones). We hear from him that he originally supposed that the functions which he has called fuchsian could *not* exist, and it was only the fact of having discovered, in his sleepless night, the op-

posite conclusion which gave his following thoughts the course they took.

That each planet moves around the sun as being attracted by it with a force proportional to the inverse square of a distance, was found by Newton to be the interpretation of the first two laws of Kepler. But there is a coefficient of proportionality—the ratio between the force of attraction and the inverse square of the distance, which ratio does not vary during the motion—and the meaning of that coefficient is to be deduced from the third law of Kepler, which concerns the comparison between the motions of different planets. The conclusion is that this coefficient is the same for all of them. All planets obey the *same* law of attraction; that conclusion does not arise from the general and synthetic view of the question, but from a precise and careful calculation. One may doubt whether Newton can have reached that last conclusion otherwise than by calculating pen in hand. Now, if the result of those calculations had been a different one, the last step of the discovery, that of identifying the force which keeps the moon revolving around the earth with the one which makes a heavy body (an apple, if we believe the legend) fall down, would not have existed.

Perhaps it is imprudent to imagine how Newton's mind functioned; but it may be noticed that the identification which he viewed required not only an algebraic but even a numerical verification, using the observed values of the magnitudes involved in the formulae (a verification which even, as is well known, was temporarily believed by Newton to be wrong), and if, strictly speaking, there could remain a doubt as to Newton's example, others are completely beyond doubt. For instance, it is certain that Georg Cantor

could not have foreseen a result of which he himself says "I see it, but I do not believe it."

In any case, moreover, the continuation of the work, just as was the case for its beginning, requires the preparation work we have spoken of. After a first stage of research has been brought to an end, the following one requires a new impulse, which can be originated and directed only when our consciousness takes account of the first *precise* result.

To take a rather familiar example, everybody understands that, intersecting two parallel straight lines by two other parallel ones, the segments thus determined are equal two by two; everybody knows that, consciously or not. But as long as it is not consciously enunciated, none of its consequences, such as similitude, can be deduced.

One of the possible cases is that the new part of the research be one which is to be carried out by exclusively conscious work, as Poincaré reports (more exactly, as I should say, by conscious work with the cooperation of fringe-consciousness); or even, as in Newton's example, one which deserves and requires a systematic and exhaustive work of that kind. To recognize such cases is again a task of our volition and the precise result is essential for that.

To sum up, every stage of the research has to be, so to speak, articulated to the following one by a result in a precise form, which I should propose to call a *relay-result* (or a *relay-formula* if it is a formula, as in Newton's interpretation of Kepler's third law). When reaching such a joining, somewhat analogous to railroad bifurcations, the new direction in which further research will follow must be decided, so that they clearly illustrate the directing action of that conscious ego which we were tempted to consider as "inferior" to unconsciousness.

The above remarks may seem to a certain extent obvious, if not childish; but it is not useless to notice that, besides the processes in the mind of any individual researchmen, they help us to understand the structure of mathematical science in general. Its improvement would have been impossible not only without verification of the results, but especially without the systematic use of what we have just called relay-results, which are very often intensely and exhaustively utilized as much as possible to the extreme end of their consequences. Such is, for instance, the role of the simple and classic fact that cutting a triangle by a parallel to one of its sides, we obtain another triangle similar to the former: a self-evident fact, but one which needed to be precisely enunciated in order to yield the long series of properties which proceed from it.