

VI. DISCOVERY AS A SYNTHESIS. THE HELP OF SIGNS

Synthesis in Discovery. Souriau, in his *Théorie de l'Invention*, writes, "Does the algebraist know what becomes of his ideas when he introduces them, in the form of signs, into his formulae? Does he follow them throughout every stage of the operations which he performs? Undoubtedly not: he immediately loses sight of them. His only concern is to put in order and to combine, according to known rules, the signs which he has before him; and he accepts with a full confidence the result thus obtained."

We have said that this author hardly seems to have gathered information from professional men. Probably, if he had, he would not have expressed himself in that way. One cannot say, however, that his statement is completely false. It can be admitted to be true, roughly speaking, as far as concerns the final phase of verifying and "precising" already mentioned in the preceding section; but, even then, things do not occur as he states. The mathematician does not so blindly confide in the results of the rules which he uses. He knows that faults of calculation are possible and even not infrequent; if the purpose of the calculation is to verify a result which unconscious or subconscious inspiration has foreseen and if this verification fails it is by no means impossible that the calculation be at first false and the inspiration be right.

If applied not to that final phase but to the total research work, the behavior which Souriau describes is that of the pupil, and even of the rather bad pupil; our efforts

aim to have him change it. The true process of thought in building up a mathematical argument is certainly rather to be compared with the process we mentioned in Section II, I mean the act of recognizing a person. An intermediate case which illustrates the analogy between the two processes is afforded by psychological studies on chess players, some of whom, as is well known, can play ten or twelve games simultaneously without seeing the chess boards. Inquiries were started, especially by Alfred Binet, in order to understand how this was possible: their results¹ may be summed up by saying that for many of these players, each game has, so to say, a kind of physiognomy, which allows him to think of it as a unique thing, however complicated it may be, just as we see the face of a man.

Now, such a phenomenon necessarily occurs in invention of any kind. We saw it mentioned in Mozart's letter (see Section I); similar statements are issued by artists like Ingres or Rodin (quoted by Delacroix, *L'Invention et le Génie*, p. 459). Only, while the happily gifted Mozart does not seem to have needed any effort in order to see the unity of his work, Rodin writes, "Till the end of his task, it is necessary for him [the sculptor] to maintain energetically, in the full light of his consciousness, his global idea, so as to reconduct unceasingly to it and closely connect with it the smallest details of his work. And this cannot be done without a very severe strain of thought."

Similarly, any mathematical argument, however complicated, must appear to me as a unique thing. I do not feel that I have understood it as long as I do not succeed in grasping it in one global idea and, unhappily, as with

¹ See, e.g., Binet's article in the *Revue des Deux Mondes*, Series 3, Vol. 117 (May-June, 1893), pp. 826-859, especially Section IV.

Rodin, this often requires a more or less painful exertion of thought.

The Use of Signs. Let us now examine a question which, as I intend to show below, is connected with the preceding one: the help which is afforded to thought by concrete representations. Such an investigation, belonging to the field of direct introspection, is possible thanks only to that fringe-consciousness which we mentioned at the end of Section II. However, we shall see that its chief results most probably also subsist in the deeper unconscious, though the latter is not directly known to us.

Words and Wordless Thought. The most classic kind of signs spoken of as cooperating with thought consists of words. Here we face a curious question on which quite divergent opinions are held.

I had a first hint of this when I read in *Le Temps* (1911): "The idea cannot be conceived otherwise than through the word and only exists by the word."² My feeling was that the ideas of the man who wrote that were of a poor quality.

But it was even more surprising for me to see such a man as Max Müller, the celebrated philologist and orientalist, maintain³ that no thought is possible without words⁴ and even write this sentence, fully unintelligible to me: "How do we know that there is a sky and that it is blue?"

² I have also seen the following topic (a deplorable subject, as far as I can judge) proposed for an examination—an elementary one, the "baccalauréat"—in philosophy in Paris: "To show that language is as necessary for us to think as it is to communicate our thoughts."

³ *Three Introductory Lectures on the Science of Thought*, delivered in London in 1887; Chicago, 1888; and also his more extensive work, *The Science of Thought*, published the same year.

⁴ It is quite possible that M. Müller's unlimited confidence in words may be due to the linguistic work of his whole life.

Should we know of a sky if we had no name for it?", admitting not only, with Herder, that "without language, man could never have come to his reason," but also adding that, without language, man could never have come even to his senses. Are animals, which do not speak, devoid of senses?

That statement of Max Müller is the more curious because he claims to find in the fact that thought is impossible without words an argument against every evolutionary theory, a proof that man cannot be descended from any animal species. The deduction, even admitting the premise, is contestable. But it could more legitimately be reversed against Max Müller's thesis if we take into account, for instance, Köhler's *Mentality of Apes*⁵ and the actions of his chimpanzees, which do imply reasoning.

Max Müller gives a historical review, which we shall reproduce in its essential parts, of opinions expressed on the question of words in thought: a review which is not devoid of interest, first in itself, next, because of the standpoint of Max Müller toward it. We hear, in the first place, that the Greeks originally used one and the same word, "logos," to denote language and thought and only later on were led to distinguish both meanings by epithets—on which, of course, the author declares them better inspired in the former case than in the latter.

Medieval scholastics, by a similitude which perhaps lies in the nature of things, agree with the beginning of Greek philosophy. Abelard, in the twelfth century, said that "Language is generated by the intellect and generates intellect." An analogous statement is to be found in a more

⁵ N.Y. 1923. See, e.g., the experiment of the "jointed stick," p. 132.

modern philosopher, Hobbes (who, generally, keeps in sympathy with the scholastics).

But, as a rule, ideas took a different course, on that subject as on many others, with the stream of thought initiated by Descartes. There is only one period in Germany, around 1900 (Humboldt, Schelling, Hegel, Herder) when philosophical minds were near to "truth," that is, to Max Müller's opinion. Hegel summarily says, "We think in nouns," as if nobody had ever doubted it.

But the other great philosophers of modern times are not so sure of the identity of language and reason. Precisely, the greatest of them—be it Locke, Leibniz or even Kant or Schopenhauer, or, more recently, John Stuart Mill—agree in a methodic doubt. Not that Leibniz does not think in words, but he does not recognize that without openly regretting it.⁶ One philosopher, Berkeley, is absolutely categorical—but in the opposite direction. He is convinced that words are the great impediment to thought.

Max Müller's passionate view of the subject leads him to qualify as "lack of courage" that general attitude of modern thinkers, which everybody else would call scientific prudence, as though no sincere opinion other than his own might exist.

Whether he admits it or not, it does exist. Immediately after the *Lectures on Science of Thought* were delivered, contradictions arose; indeed they came from most various parts.⁷ Above all, there came the authorized voice of another first-rank scholar, Francis Galton, the great ge-

⁶ *Dialogue on the Connection Between Things and Words*: "It troubles me greatly [*Hoc unum me male habet*] to find that I can never acknowledge, discover or prove any truth except by using in my mind words or other things."

⁷ See the exchange of letters at the end of the *Introductory Lectures*.

neticist, who, moreover, after having begun as an explorer, has done important work in psychological matters. The latter's great habit of introspection allows him to assert that his mind does not behave at all in the way supposed by Max Müller to be the only possible one. Whether he is playing billiards and calculating the course of his ball or investigating higher and more abstract questions, his thought is never accompanied by words.

Galton adds that he sometimes happens, while engaged in thinking, to catch an accompaniment of *nonsense* words, just "as the notes of a song might accompany thought." Of course, nonsense words are something quite different from real words; we shall see later to what kind of images they may be reasonably compared.

That disposition of mind in Galton is not devoid of inconvenience for him. "It is," he says, "a serious drawback to me in writing, and still more in explaining myself, that I do not so easily think in words as otherwise. It often happens that after being hard at work, and having arrived at results that are perfectly clear and satisfactory to myself, when I try to express them in language I feel that I must begin by putting myself upon quite another intellectual plane. I have to translate my thoughts into a language that does not run very evenly with them. I therefore waste a vast deal of time in seeking for appropriate words and phrases, and am conscious, when required to speak on a sudden, of being often very obscure through mere verbal maladroitness, and not through want of clearness of perception. That is one of the small annoyances of my life."

I have wanted to reproduce at length that statement of Galton, because in his case I exactly recognize mine, in-

cluding the rather regrettable consequence which I experience just as he does.

The fact that it is impossible for Max Müller to recall lightning without thinking of its name does not mean that "we" are unable to do so. As for myself, if I remember lightning, I see in my mind the flash of light which I have admired several times, and I should need an instant of reflection—a short one, of course, but certainly an instant—if I should wish the corresponding word to recur to me. Just as for Galton, such a translation from thought to language always requires on my part a more or less difficult effort. Whether the verses of Boileau

"Ce qui se conçoit bien s'énonce clairement
Et les mots pour le dire arrivent aisément,"

are justified or not concerning other people, it is certain that they are not true for me. I have a tangible proof of that—an "objective" one, I could say—in the fact that it is difficult for me to deliver a lecture on anything but mathematical subjects without having written down practically every part of it, the only means of avoiding constant and painful hesitation in the expression of thought which is very clear in my mind.

Galton legitimately points out how strange it is that Max Müller has utterly failed to understand that other people's minds may be different from his own: a most common error, but one which it is surprising to find among men accustomed to psychological studies. Differences between minds being, on the contrary, undeniable according to what we have just found, the question ought to be settled not by polemics but by inquiries relating to every human race and every class of men and, if possible (we

shall see that there may be some difficulty in this), not only among intellectual people. Galton, inquiring, as he says, as much as occasion has allowed him, finds a certain percentage, though a minor one, of persons whose thought is habitually carried on without the use of mental or spoken words. One may wonder that a man as well acquainted with statistical operations as Galton does not give a precise percentage; a possible reason for that will appear below.⁸

Mental Pictures in Usual Thought. Thought can be accompanied by concrete representations other than words. Aristotle admitted that we cannot think without images. Taine's well-known work *On Intelligence* is chiefly devoted to the importance, in the constitution of ideas, of images, which he defines, at the beginning of his Volume II, as recurring, surviving and spontaneously resurging sensations. However, he is now believed to have exaggerated that importance and described it as a too exclusive one.

At about the same time, Alfred Binet was making an important improvement in the study of that question by attacking it in the experimental way.⁹ He investigated some twenty persons, but chiefly two young girls (aged thirteen and fourteen) in his own family, whose valuable help in somewhat delicate psychological researches, at such a youthful age, is a very remarkable thing. He submits them sometimes to pure experiments, but more often to experiment combined with introspection. For instance, asking a question or pronouncing a word, he inquires what ideas, images, etc., this has suggested to the subject. The

⁸ Galton, in his *Inquiries into Human Faculty*, has conducted inquiries, interpreted according to the rules of statistics, on mental imagery considered in itself. The question for us would be to have a similar inquest on imagery as a help to thought.

⁹ *Etude Expérimentale de l'Intelligence*, Paris (1903).

method has been criticized and indeed there is the objection which can be advanced against almost every kind of psychological experiment, viz., an involuntary suggestion from the experimenter himself. This, however, is not to be feared when the results are of an unexpected nature as some have been for Binet. As a matter of fact, Binet's method is considered by psychologists as not being invalidated by that or analogous objections, to which convincing answers have been given by Bühler;¹⁰ and a somewhat similar method was used, a little later, by the so-called Wurtzbourg school. Its creation, however, belongs to Binet.

In Binet's experiments, the question of words is dealt with incidentally. The answer on that point is favorable to Galton against Max Müller. To one of the girls,¹¹ an answer in words appears as "an image which cuts thought." Thought is something which appears to her suddenly like any kind of feeling.

What is more unexpected is that even the intervention of images is minimized, contrary to Taine's theory. The precision of the answer is striking.¹² "In order to get images, I must no longer have anything to think of. They [ideas and images] are separated from each other and never come together. I never have any images when a word suggests to me a very great number of thoughts. I must wait for a while. When, with respect to this word, I have exhausted every thought, then images come and if thoughts begin again, the images fade, and alternate."

On that point, Binet himself concludes: "Later on, I

¹⁰ *Archiv f. die Ges. Psych.*, Vol. IX (1907); Vol. XII (1908) especially pp. 98-128. See G. Dumas' *Traité de Psychologie*, Vol. I, Chap. IV and Vol. II, pp. 113 ff.

¹¹ *Etude Expérimentale de l'Intelligence*, p. 107.

¹² *ibid.*, p. 124.

was able to convince myself that Armande was quite right; I admit that there exists a kind of antagonism between image and reflection, the more so when the image is very intense. It is in reverie and dream that the finest images arise." There is also the fact, observed by Galton and others, that women and children have finer images than adult men who are superior in reflection.

Later experiments by Dwelshauvers (*Les Mécanismes Subconscients*), carried on with students, led to the same main conclusions as Binet's about the conditions of the apparition of images. He finds that images appear only if we give our ideas uncontrolled freedom, i.e., when we are dreaming while awake. As soon as full consciousness, voluntary consciousness, returns, images weaken, darken; they seem to withdraw into some unknown region.

Mental Pictures in Tense Thought. More recent authors (Delacroix, James Angell, Titchener, Varendonck, etc.) have also treated that same subject of words and images in thought. Most of their works, however, will not directly occupy us, on account of a distinction which is especially necessary in our subject.

Psychologists have already distinguished between two kinds of thought. There is "free" thought, which takes place when you let your thoughts wander, without directing them toward any special goal; and there is "controlled" thought, when such a direction is given.¹³ The second term is not precise enough for our purpose. There is already a direction in your thought when you are asked what is the date; but the case of inventive thought is obviously differ-

¹³ R. S. Woodworth, *Psychology* (4th edition, 1940), p. 33. However, Woodworth speaks of your being asked a *difficult* question, which would mean our third case of the tense thought, rather than the simply controlled one.

ent. It requires a certain effort of concentration; it is not only controlled, but tense.

There is no reason why the processes of those three kinds of thought should be the same; and actually they are not. The last case is the only one which directly concerns us.

Binet's View. As a conclusion to his series of experiments, Binet is¹⁴ inclined to think that words or sensorial images may be useful for giving a precise form to feelings or thoughts which, without both these helps, would remain too vague; even, to give us a full consciousness of that thought which would, otherwise, remain an unconscious act of mind; in order to allow the passage of ideas from unconscious to conscious: more precisely, from the unconscious where they are somewhat vague to consciousness where they acquire precision.

I was myself inclined, for a while, to admit that conception of Binet. Indeed, it satisfied to a certain extent the double and seemingly contradictory condition:

(a) That the help of images is absolutely necessary for conducting my thought.

(b) That I am never deceived and even never fear to be deceived by them.

However, further reflection led me to a different conception. Indeed, the case of Binet's or Dwelshauvers' experiments is not ours; it deals with a controlled thought, but not with a tense one. The two girls are asked such questions as, "What appears in your mind when you think of what you did yesterday?" The most difficult question, as far as I could see on examining Binet's book, was, "Think of what you would like to do if you could remain three hours by yourself, being completely free in your actions?"

¹⁴ *Etude Expérimentale de l'Intelligence*, p. 108.

Personal Observations. The case of research work is, of course, very different, for which reason I have wished to understand what takes place in my own mind when I undertake to build up or to comprehend—I have said in the beginning that there is no essential difference—a mathematical argument.

I insist that words are totally absent from my mind when I really think and I shall completely align my case with Galton's in the sense that even after reading or hearing a question, every word disappears at the very moment I am beginning to think it over; words do not reappear in my consciousness¹⁵ before I have accomplished or given up the research, just as happened to Galton; and I fully agree with Schopenhauer when he writes, "Thoughts die the moment they are embodied by words."

I think it also essential to emphasize that I behave in this way not only about words, but even about algebraic signs. I use them when dealing with easy calculations; but whenever the matter looks more difficult, they become too heavy

¹⁵ It is quite possible, and rather probable, that words are present in fringe-consciousness. Such is the case, I imagine, for me as concerns words used in mathematics. I doubt, however, that it is so for some other kind of thought because, if it were, I should have less difficulty in finding them. An evident misunderstanding as to the meaning of the question occurs when, after citing William Hamilton who observes that "a cognition must have been already there before it could receive a sign," so that the idea must necessarily precede the word, Max Müller claims to be in agreement (*sic*) with him, because William Hamilton's statement means an "almost contemporaneous" progress of thinking and naming. Though being only an occasional psychologist, I know enough to understand that mental processes are often rapid ones and that it would be absurd to study them without distinguishing between "almost contemporaneous" and simultaneous states. Besides, William Hamilton expresses his feeling on the question in a striking way, saying: "Speech is thus not the mother, but the godmother of knowledge."

a *baggage* for me. I use concrete representations, but of a quite different nature.

One example of this kind is already known in the history of science. It was given by Euler, in order to explain to a Swedish princess the properties of syllogism. He represents general ideas by circles; then if we are to think of two categories of things, A and B, such that every A is a B, we shall imagine a circle A lying inside a circle B. If, on the contrary, we are told that no A is a B, we shall imagine the circle A as lying completely outside B; while if some A's are B's and some not, the two circles ought to be intersecting ones. Now, personally, if I had to think of any syllogism, I should not think of it in terms of words—words would hardly allow me to see whether the syllogism would be right or wrong—but with a representation analogous to Euler's, only not using circles, but spots of an undefined form, no precise shape being necessary for me to think of spots lying inside or outside of each other.

To consider a slightly less simple case, let us take an elementary and well-known proof in arithmetic, the theorem to be proved being: "The sequel of prime numbers is unlimited." I shall repeat the successive steps of the classic proof of that theorem, writing, opposite each of them, the corresponding mental picture in my mind. We have, for instance, to prove that there is a prime greater than 11.

STEPS IN THE PROOF

MY MENTAL PICTURES

I consider all primes from 2 to 11, say 2, 3, 5, 7, 11.

I see a confused mass.

I form their product

$$2 \times 3 \times 5 \times 7 \times 11 = N$$

N being a rather large number, I imagine a point rather remote from the confused mass.

<p>I increase that product by 1, say N plus 1.</p> <p>That number, if not a prime, must admit of a prime divi- sor, which is the required number.</p>	<p>I see a second point a little beyond the first.</p> <p>I see a place somewhere be- tween the confused mass and the first point.</p>
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What may be the use of such a strange and cloudy imagery? Certainly, it is not meant to remind me of any property of divisibility, prime numbers and so on. This is most important because any such information which it could give me would be likely to be more or less inaccurate and to deceive me. Thus, that mechanism satisfies condition (b) previously required. On the contrary, this condition is but partly satisfied by Binet's hypothesis: giving precision to unconscious ideas would always risk altering them.

But at the same time, one can easily realize how such a mechanism or an analogous one may be necessary to me for the understanding of the above proof. I need it in order to have a simultaneous view of all elements of the argument, to hold them together, to make a whole of them—in short, to achieve that synthesis which we spoke of in the beginning of this section and give the problem its physiognomy. It does not inform me on any link of the argument (i.e., on any property of divisibility or primes); but it reminds me how these links are to be brought together. If we still follow Poincaré's comparison, that imagery is necessary in order that the useful hookings, once obtained, may not get lost.

Indeed, every mathematical research compels me to build such a schema, which is always and must be of a vague character, so as not to be deceptive. I shall give one less elementary example from my first researches (my thesis). I

had to consider a sum of an infinite number of terms, intending to valuate its order of magnitude. In that case, there is a group of terms which chances to be predominant, all others having a negligible influence. Now, when I think of that question, I see not the formula itself, but the place it would take if written: a kind of ribbon, which is thicker or darker at the place corresponding to the possibly important terms; or (at other moments), I see something like a formula, but by no means a legible one, as I should see it (being strongly long-sighted) if I had no eye-glasses on, with letters seeming rather more apparent (though still *not legible*) at the place which is supposed to be the important one.

I have been told by some friends that I have a special way of looking when indulging in mathematical research. I hardly doubt that this especially accompanies the construction of the schema in question.

This is in connection with the question of intellectual fatigue. I have asked some prominent physiologists, especially Louis Lapicque, how intellectual work can produce fatigue, as no "work," in the physicist's meaning of the word, seems to be produced. Lapicque's opinion is that intellectual work ought to be comparable to nothing more than the act of turning the pages of a book. However, intellectual fatigue exists: from the objective and physiological point of view, it has been studied in an important book of Binet and Victor Henri. That part of the question is beyond my sphere. From the psychological point of view, it can be considered as certain that, similarly to what we have said of Rodin's case, fatigue corresponds to the effort for synthesis, to the fact of giving the research its unity and,

therefore, at least in my case, to the constitution of a proper schema.

One or two observations can be added:

If I should use a blackboard and write the expression $2 \times 3 \times 5 \times 7 \times 11$, the above described schema would disappear from my mind as having obviously become useless, and would be automatically replaced by the formula which I should have before my eyes.

Then I must observe that I distinctly belong to the auditory type;¹⁶ and precisely on that account my mental pictures are exclusively visual. The reason for that is quite clear to me: such visual pictures are more naturally vague, as we have seen it to be necessary in order to lead me without misleading me.

I also add that the case we have just examined especially concerns arithmetical, algebraic or analytic studies. When I undertake some geometrical research, I have generally a mental view of the diagram itself, though generally an inadequate or incomplete one, in spite of which it affords the necessary synthesis—a tendency which, it would ap-

¹⁶ I have a rather bad memory of physiognomies and am much exposed to failures of recognition or false recognitions; on the contrary, I am very sensitive to the sound of names, being much more desirous to see such rivers as the Mohawk or the Mattawamkeag than even their beauty would warrant, because their mere names call to my mind the idea of forests and Indian life. Also, I am rather less sensible than others to likenesses in faces and rather more sensible to likenesses in voices.

Many mistakes by automatic writing (see Section II, p. 22) are due to auditory mental images. Examples of that kind are classic. Personally, they are frequent in my case: my conscious ego dictates to my unconscious one, which writes a word instead of another one if their sounds are somewhat alike. While writing the present work, I remember to have written "simple" instead of "same place" and "will she" instead of "we shall." I should think that such auditive mistakes are more frequent on my part when I am writing in English than in French—quite naturally.

pear, results from a training which goes back to my very earliest childhood.

Paradoxical as it seems, it very often happens that, in those geometrical problems, I use successfully a process quite opposite to the synthesis I have explained in what precedes. I happen to abstract some special part of the diagram and consider it apart from the rest, this consideration leading to a "relay-result." The whole argument is, nevertheless, even in that case, grasped as a unique entity, as a synthesis into which such a relay-result, if existent, is included. That is a process which, according to Pierre Boutroux¹⁷ (see below), Descartes says to be frequent in Greek geometry.

Respective Roles of Full Consciousness and Fringe-Consciousness. The above observations concern the functioning of thought when it is intensely concentrated, be it in an entirely conscious work or in a conscious preliminary work.

Now, as we have explained at the end of Section II, that very concentration gives us a possibility of distinguishing between full consciousness and fringe-consciousness, a distinction which is rather difficult in other circumstances, but which, in that case, is rather easily accessible to observation.

What does observation give as to the phenomena just described?

It could be supposed a priori that the links of the argument exist in full consciousness, the corresponding images being thought of by the subconscious. My personal introspection undoubtedly leads me to the contrary conclusion: my consciousness is focused on the successive images, or more exactly, on the global image; the arguments themselves wait, so to speak, in the antechamber (see p. 25) to

¹⁷ P. Boutroux, however, gives no precise reference.

be introduced at the beginning of the "precising" phase.

This instance most clearly illustrates the nature and role of the fringe-consciousness, which is, so to speak, at the service of full consciousness, being ready to appear in it whenever wanted.

Other Stages of Research. What happens when there is a period of incubation: in other words, an action of the deeper unconscious? Of course, no direct answer is available; but a strong presumption that there is some mechanism of an analogous kind at work results from the fact that it seems to be the fittest one to satisfy the double condition (a), (b) which is to be fulfilled.

I should even interpret in a similar way the case of illumination. When I think of the example mentioned in Section I (see page 8) I see a schematic diagram: a square of whose sides only the verticals are drawn and, inside of it, four points being the vertices of a rectangle and joined by (hardly apparent) diagonals—a diagram the symbolic meaning of which will be clear for technicians. It even seems to me that such was my visualization of the question in 1892, as far as I can recollect. Of course, remembrances going back half a century are not quite reliable; however, we recognized symbolic diagrams as being essential to a synthetic view of questions, and it seems to me that such a synthetic view is at least as necessary in cases of illumination as in conscious work. If we admit this line of reasoning, illumination would be transmitted from a lesser or greater depth of unconsciousness to fringe-consciousness, which would have it represented by a symbolic diagram in the conscious ego.

That image and its meaning are in some way connected and, at the same time, independent, is observed by Watt,

Archiv. f.d. Ges. Psych., 1904, Vol. IV (see G. Dumas' *Traité de Psychologie*, Vol. I, Chap. IV). It seems to me that such a kind of simultaneous connection and independence is enlightened by the intervention of fringe-consciousness.

Then comes the verifying and "precising" stage. In that final phase of the work, I may use algebraic symbols; but, rather often, I do not use them in the usual and regular way. I do not take time to write the equations completely, only caring to see, so to speak, how they look. These equations, or some terms of them, are often disposed in a peculiar and funny order like actors in a scenario, by means of which they "speak" to me, as long as I continue to consider them. But if, after having been interrupted in my calculations, I resume them on the following day, what I have written in that way is as if "dead" for me. Generally, I can do nothing else than throw the sheet away and begin everything anew, except if, in the first day, I have obtained one or two formulae which I have fully verified and can use as relay-formulae.

As to words, they remain absolutely absent from my mind until I come to the moment of communicating the results in written or oral form, or (very exceptionally) for relay-results; in the latter case, they may, as William Hamilton observes, be the intermediary "necessary to give stability to our intellectual process, to establish each step in our advance as a new starting point for our advance to another beyond"—in which William Hamilton is right but for the fact that any relay-result can play such a role.¹⁸

¹⁸ William Hamilton uses an interesting comparison to the process of tunneling through a sand bank. "In this operation, it is impossible to succeed unless every foot, nay, almost every inch in our progress be

Another Conception. After having acquired some information about the behaviorist school, I wondered how behaviorist doctrine dealt with our present question and whether it agreed with my observations. I understand that, for behaviorism, we do not necessarily think in words, but that, otherwise, our thought may consist of muscular motions, such as a shrug of the shoulders, motions of the eyelids or the eyes, etc.

I have no recollection of such kind of motions connected with my research work. Of course, I cannot watch my motions while deeply engaged in research, but witnesses of my daily life and work can assert that they never saw anything of that kind. They have only observed the special "inside" look which I often happen to have when plunged in deeply concentrated reflection. What I can say is that I do not see what kind of motions could help me to get a clear view of more or less difficult reasonings, while, on the contrary, we have seen that proper mental pictures can evidently be helpful for that.

An Inquiry among Mathematicians. It is natural to investigate, with respect to our present subject, the behavior of mathematicians in general. Unhappily, I was unable to learn about French mathematicians, having thought of the question only after my departure from Europe.

About the mathematicians born or resident in America, whom I asked, phenomena are mostly analogous to those secured by an arch of masonry before we attempt the excavation of another. Now, language is to the mind precisely what the arch is to the tunnel. The power of thinking and the power of excavation are not dependent on the words in the one case, on the mason-work in the other; but without these subsidiaries, neither process could be carried on beyond its rudimentary commencement."

More generally, the function thus described belongs to what we have called relay-results. In inventive effort, those do not always imply words.

which I have noticed in my own case.¹⁹ Practically all of them—contrary to what occasional inquiries had suggested to Galton as to the man in the street—avoid not only the use of mental words but also, just as I do, the mental use of algebraic or any other precise signs; also as in my case, they use vague images. There are two or three exceptional cases, the most important of which is the mathematician George D. Birkhoff, one of the greatest in the world, who is accustomed to visualize algebraic symbols and to work with them mentally. Norbert Wiener's answer is that he happens to think either with or without words. Jessie Douglas generally thinks without words or algebraic signs. Eventually, his research thought is in connection with words, but only with their rhythm, a kind of Morse language where only the numbers of syllables of some words appear. This, of course, has nothing in common with Max Müller's thesis and is rather analogous to Galton's use of meaningless words.

G. Pólya's case—I intend to speak only of men who have made quite significant discoveries—is different. He does make an eventual use of words. "I believe," he writes to me, "that the decisive idea which brings the solution of a problem is rather often connected with a well-turned word or sentence. The word or the sentence enlightens the situation, gives things, as you say, a physiognomy. It can precede by little the decisive idea or follow on it immediately; perhaps, it arises at the same time as the decisive idea. . . . The right word, the subtly appropriate word, helps us to recall the mathematical idea, perhaps less completely and less objectively than a diagram or a mathe-

¹⁹ At the moment of printing, a letter from Professor Einstein reaches me, containing information of capital interest. See Appendix II.

mathematical notation, but in an analogous way. . . . It may contribute to fix it in the mind." Moreover, he finds that a proper notation—that is, a properly chosen letter to denote a mathematical quantity—can give him similar help; and some kind of puns, whether of good or poor quality, may be useful for that purpose. For instance, Pólya, teaching in German at a Swiss university, usually made his junior students observe that z and w are the initials of the German words "Zahl" and "Wert," which precisely denote the respective roles which z and w had to play in the theory which he was explaining.

That case of Pólya seems to be quite exceptional (I did not meet with any similar one among the other men who answered me).²⁰ Even he, however, does not use words as equivalents of ideas, since he uses *one* word or one or two letters to symbolize a whole line of thought; his psychological process would be in agreement with Stanley's statement²¹ that "language, as an indicator, can only indicate by suggesting to our consciousness what is indicated, as object, thought or feeling, even in most summary and unself-conscious form to which it is brought by practice."

The mental pictures of the mathematicians whose answers I have received are most frequently visual, but they may also be of another kind—for instance, kinetic. There can also be auditive ones, but even these, as the example of J. Douglas shows, quite generally keep their vague character.²² For B. O. Koopman, "images have a symbolic rather

²⁰ I have just heard of the rather analogous case of Professor Chevalley.

²¹ *Psychological Review*, Vol. IV (1891), p. 71. In that place, Stanley chiefly deals with poetical invention, where the role of words is evidently more important than elsewhere.

²² One of my colleagues of Columbia writes to me that his mathematical thought is usually accompanied by visual images and hardly ever by words other than vague exclamations of surprise, irritation, elation, etc.

than a diagrammatic relation to the mathematical ideas" which are considered, a description whose analogy with the above is evident. Professor Koopman's observations also agree with mine on the fact that such images appear in full consciousness while the corresponding arguments provisionally remain in the "antechamber."

We can say as much of the observations which Ribot²³ has gathered by questioning mathematicians. Some of them have told him that they think in a purely algebraic way, with the help of signs; others *always* need a "figured representation," a "construction," even if this is "considered as pure fiction."

Some Ideas of Descartes. In the *Regulae ad Directionem Ingenii*, which, in their second half (from the 14th rule on), deal with the role of imagination in science, Descartes seems to have conceived the idea of processes similar to those we speak of. At least, this can be induced from some places of the analysis of the *Regulae* made by Pierre Boutroux.²⁴ For instance, he is reported by Pierre Boutroux to have said that "Imagination, by itself, is unable to create Science, but we must, in certain cases have recourse to it. First, by focusing it on the object we want to consider, we prevent it from going astray and, moreover, it can be useful in awakening within us certain ideas." Again, "Imagination will chiefly be of great use in solving a problem by several deductions, the results of which need to be coordinated after a complete enumeration. Memory is necessary to retain the data of the problem if we do not use them all from the beginning. We should risk forgetting them if the

²³ *Evolution des Idées Générales*, p. 143.

²⁴ *L'Imagination et les Mathématiques selon Descartes*, Bibl. de la Faculté des Lettres de Paris, Vol. 10 (1900).

image of the objects under consideration were not constantly present to our mind and did not offer all of them to us at each instant."

This is the role of images as we described it above. However, Descartes distrusts that intervention of imagination and wishes to eliminate it completely from science. He even reproaches ancient geometry for having used it. He wants to eliminate imagination from every branch of science by reducing all of them to mathematics (which he tried to do, but did not succeed in doing), mathematics consisting, more than any other science, of pure understanding.

To see what we must think of such an idea, we need only recall how Descartes' program has been carried out by modern mathematicians. First, as is well known, geometry can be completely reduced to numerical combinations by the help of analytical geometry which Descartes himself created. But we have just seen that deductions in the realm of numbers may be, at least in several mathematical minds, most generally accompanied by images.

More recently, another rigorous treatment of the principles of geometry, which, logically speaking, has been fully freed from any appeal to intuition, has been developed on quite different bases by the celebrated mathematician Hilbert. His beginning, which is now classic among mathematicians, is "Let us consider three systems of things. The things composing the first system, we will call *points*; those of the second, we will call *straight lines*, and those of the third system, we will call *planes*," clearly meaning that we ought by no means to inquire what those "things" may represent.

Logically, of course—and this is all that is essential—the result announced is fully attained and every interven-

tion of geometrical sense eliminated: that is, theoretically unnecessary to follow the reasoning from the beginning to the end. Is it the same from the psychological point of view? Certainly not. There is no doubt that Hilbert, in working out his *Principles of Geometry*, has been constantly guided by his geometrical sense. If anybody could doubt that (which no mathematician will), he ought simply to cast one glance at Hilbert's book. Diagrams appear at practically every page. They do not hamper mathematical readers in ascertaining that, logically speaking, no concrete picture is needed.²⁵

This is again a case where one is guided by images without being enslaved by them, and it is again possible (at least in my own case) thanks to the same division of work between proper consciousness and fringe-consciousness.²⁶

Similarly, Descartes censures the habit noticed by him in Greek geometers (see above) of considering separately one part of a diagram. There is no reason for that objection. We meet with the same confusion between logical and psychological processes. The method in question no more compromises the rigor of the argument than the image mentioned above compromises the proof of the fact that prime numbers form an unlimited sequel.

²⁵ There is already a paradox, as Klein notices, in the fact that we can reason on an angle equal to the millionth part of a second, though we are completely unable to distinguish between the sides of such an angle; and its discussion (*Revue de Métaphysique et de Morale* [1908], p. 923) by Winter, one of the philosophers who had the best understanding of scientific subjects, shows the analogy of that circumstance with our observations in the text.

²⁶ Another instance, which we shall meet in Section VII (see p. 108 note) will illustrate this even more clearly and convincingly. Indeed, in the latter instance of Section VII, no doubt can subsist (at least as concerns my own mind) on the way the division of work mentioned in the text takes place.

Other Thinkers. We do not have many data on that question in fields other than the mathematical one. It is curious that, according to the above-mentioned work of Binet (*Etude Expérimentale de l'Intelligence*, pp. 127-129), even in free thought, vague images may occur as representatives of more precise ideas.

An instance quite analogous to our above description is that of the economist Sidgwick, which he himself reported at the International Congress of Experimental Psychology, in 1892. His reasonings on economic questions were almost always accompanied by images, and "the images were often curiously arbitrary and sometimes almost undecipherably symbolic. For example, it took him a long time to discover that an odd, symbolic image which accompanied the word 'value' was a faint, partial image of a man putting something on a scale." Also a most curious process occurs among musical composers, according to Julius Bahle (*Der Musikalische Schaffensprozess*, Leipzig, Hirzel, 1936; quoted by Delacroix, *L'Invention et le Génie*, p. 520). Several of them see their creations in their initial conception, in a visual form (inspiration is what he calls a "Tonvision"). One of them perceives in that way, without any precise musical presentation, "the main line and main characteristics of his music. Besides, it is perhaps rather difficult to say to what extent music is absent from that formal schema."²⁷

I have asked only a few men belonging to other branches of mental activity. The answers have been various and I

²⁷ A painter tells me that, in the first phase of composition, his visual images are voluntarily vague.

cannot assert that results could not differ from our previous ones.²⁸

Some scientists have told me of mental pictures quite analogous to those which we have described. For instance,²⁹ Professor Claude Levi-Strauss, when thinking about a difficult question in his ethnographic studies, sees, as I do, unprecise and schematic pictures which, moreover, have the remarkable character of being three-dimensional. Also, asking a few chemists, all of them reported absolutely wordless thought, with the help of mental pictures.

The physiologist André Mayer's mind behaves quite differently. He tells me that his thought immediately appears to him in a fully formulated form, so that no effort whatever is necessary to him in order to write it down.

It would be interesting to know how medical men behave, in that respect, in the difficult act of arriving at a diagnosis. I had the opportunity of asking a prominent one, and he told me that he thinks without words in that case, though his thought uses words in theoretical and scientific studies.

A type of thinking which seems surprising at first has been discovered by the psychologist Ribot,³⁰ who finds it

²⁸ Our statesman Aristide Briand, according to what was observed by one of his closest collaborators who was able to see him often at work, did not think in words when he planned his speeches. Words appeared only at the very moment of pronouncing them.

It would certainly be worth while to know about the opinion of some important military leaders. There could be no case where a simultaneous view of the synthesis and of every detail would be more essential.

²⁹ Such is also the case of Professor Roman Jakobson (see p. 96).

³⁰ *L'Evolution des Idées Générales*, p. 143. Jean Perrin, according to information given to me by his son, had *intermittently* pictures of the typographic-visual type; Francis Perrin generally thinks without words; but, from time to time, a word appears to him. Sidgwick's ideas appeared in typographic-visual representations when, instead of economic subjects, he was thinking of mathematics or logic.

to be more frequent than would be expected. It is what he calls the "typographic visual type" and consists in seeing mentally ideas in the form of corresponding printed words. The first discovery of this by Ribot was the case of a man whom he mentions as a well-known physiologist. For that man, even the words "dog, animal" (while he was living among dogs and experimenting on them daily) were not accompanied by any image, but were seen by him as being printed. Similarly, when he heard the name of an intimate friend, he saw it printed and had to make an effort to see the image of this friend. It was the same with the word "water," and carbonic acid or hydrogen appeared to his mind either by their printed full names or by their printed chemical symbol. Being strongly surprised by that statement, the sincerity and accuracy of which were not to be doubted, Ribot later on observed that that case was by no means a unique one and similar ones were to be found in several people.

Moreover, according to Ribot, men belonging to that typographic-visual type cannot conceive how other people's thought can proceed differently.

This is the state of mind which we have already noticed in Max Müller himself as concerns, more generally, thought in words and which is really stupefying when we find it among men accustomed to dealing with philosophical matters. How can we wonder that people have been burned alive on account of differences in theological opinions, when we see that a first-rate man like Max Müller, apropos of a harmless question of psychology, uses scornful words toward his old master Lotze, for having written that the

logical meaning of a given proposition is in itself independent of the form in which language expresses it.³¹

Thus, we have let ourselves be induced to deal with a chapter of psychology rather different from the one which is our main object. Some parts of this section could be called "A case of psychological incomprehension."

This is not at all the only instance of the double fact: (1) that the psychology of different individuals may differ in some essential points; (2) that, if so, it may be almost impossible for the one to conceive the state of mind of the other.³²

Is Thought in words without inconvenience? Of course, I must myself be on my guard against the same lack of

³¹ Max Müller concedes that he can "with some effort" enter into the mind of a decided adversary like Berkeley, "a kind of philosophical hallucination," to use his own words. But he cannot understand the opinion that *most* of our thoughts are carried on in language but not quite all, or that *most* people think in words but not all. That some of the greatest writers may have said that not from "lack of courage" but because facts are like that, visibly lies beyond his imagination.

³² Paradoxical as it seems, there are two such instances in the domain of mathematics. Some years before World War I, a question which, though a mathematical one, was contiguous to metaphysics raised a lively discussion among some of us, especially between myself and one of my best and most respected friends, the great scientist Lebesgue. We could not avoid the conclusion that evidence—that starting point of certitude in every order of thinking—did not have the same meaning for him and for me. Only, of course, we were never tempted to despise each other merely because we recognized the impossibility of understanding each other.

The subject in question belonged to the theory of "sets." Now, when, in 1879-1884, Georg Cantor communicated his fundamental results on that theory (now one of the bases of contemporary science), one of them looked so paradoxical and upset so radically all our fundamental notions that it unleashed the decided hostility of Kroneker, one of the leading mathematicians in that time, who prevented Cantor from getting any new appointment in German universities and even from having any memoir published in German periodicals. Of course, the proof of that result is as clear and rigorous as any other proof in mathematics, leaving no possibility of not admitting it.

comprehension. Certainly, I must confess my incomprehension of the fact that the typographic-visual type or other verbal types are possible, and I can hardly refrain from thinking of Goethe's verses:

“Denn wo Begriffe fehlen,
Da stellt ein Wort zur rechten Zeit sich ein.”

But I cannot forget that such men as Max Müller and others, not mediocre, think in such a way, though I do not succeed in understanding it. In this connection, I regret that Ribot has not published the name of the physiologist he speaks of, and that we are therefore unable to form any opinion of the value of his work.

For those of us who do not think in words, the chief difficulty in understanding those who do lies in our inability to understand how they can be sure they are not misled by the words they use—see our condition (b), page 74. As Ribot says,³³ “The word much resembles paper money (banknotes, checks, etc.), having the same usefulness and the same dangers.”

Such a danger has not remained unnoticed. Locke mentions many men as using words *instead* of ideas, and we have seen that Leibniz cannot help experiencing a certain anxiety as to the influence of thinking in words on the course of his thought.

Curiously enough, Max Müller himself says that indirectly. To Kant, he opposes his friend Hamann, whom he

³³ *The Psychology of Attention*, p. 52 (translation of 1911; p. 85 of the French edition of 1889). Ribot, in that same place, also describes the evolution of that function of the word. He writes, “Learning how to count in the case of children, and, better still, in the case of savages, clearly shows how the word, at first firmly clinging to objects, then to images, progressively detaches itself from them, to live an independent life of its own.”

highly praises, and he cites the latter as having written "Language is not only the foundation for the whole faculty of thinking, but the central point also from which proceed the misunderstandings of reason by herself. . . . The question with me is not what is the reason, but what is the language? And here, I suspect, is the ground of the paralogisms and antinomies with which the reason is charged."

This would be all right if, consequently, Max Müller warned us to beware of such misunderstandings caused by language; but on the contrary, he maintains that words, by themselves, could never produce any error. "The word itself is clear and simple and right; we ourselves only derange and huddle and muddle it."

I should not have spoken once more of Max Müller if the statement expressed in that passage of the *Introductory Lectures* did not go even beyond the hitherto examined question of words in thought. Immediately after that citation of Hamann, apparently yielding to a professional deformation, he speaks to us of "the science of thought, founded as it is on the science of language." Will he have us believe that language not only must accompany thought, but must govern it?

Unhappily for his thesis, such a tendency seems not to have been always harmless to him. It requires a man who identifies thought with words to attack Darwin's theory³⁴ by only taking account of the word "selection" and neglecting, as allegedly a "metaphorical disguise," the meaning of it in Darwin's ideas.

The thinker using mental words may, on the contrary, understand that not only words, but every kind of auxiliary

³⁴ *The Science of Thought*, Vol. I, p. 97.

signs only play the role of kinds of labels attached to ideas. He will, more or less consciously, apply proper methods (about which it would be interesting to inquire) in order to give them that role and no other one. We have seen that Pólya himself, the only one among the mathematicians I have consulted who makes so much use of words, introduces a single one in a whole course of thought so as to remind himself of a central idea, while Jessie Douglas represents some of them by their mere syllabic rhythm. Similarly, one of my colleagues in literary matters thinks in words, but from time to time introduces a *nonexistent* word. This process, to be compared with Jessie Douglas's or Galton's, seems to me to be evidently useful for the same purpose.

In the thought of Leibniz, we can be sure that such misunderstandings as were dreaded by Hamann could not occur: first, because he is Leibniz, then, because he is aware of the danger. But, though being little acquainted with the theories of metaphysicians, I am rather disquieted on reading in Ribot's *Evolution des Idées Générales* that among them the typographic-visual type seems to be overwhelmingly the most frequent one.

Among philosophers, there seems, indeed, to be a certain tendency to confuse logical thought with the use of words. For instance, it is difficult not to recognize it in William James when he complains³⁵ that, "we are so subject to the philosophical tradition which treats *logos* or discursive thought generally as the sole avenue to truth, that to fall back on raw un verbalized life has more of a revealer, etc."—the word "un verbalized" hardly leaving any doubt that he has used the word *logos* in the ancient Greek sense.

Is not that tendency likely eventually to mislead those

³⁵ *A Pluralistic Universe*, p. 272.

who let themselves be governed by it? Reading Fouillée's objections concerning the unconscious in his *Evolutionisme des Idées-Forces* (see Section II, p. 28), I wonder whether he does not mistake words for reasons.

I even feel some uneasiness when I see that Locke and similarly John Stuart Mill consider the use of words necessary whenever complex ideas are implied. I think, on the contrary, and so will a majority of scientific men, that the more complicated and difficult a question is, the more we distrust words, the more we feel we must control that dangerous ally and its sometimes treacherous precision.

A Valuable Description. Though, in the question of words in thought, divergent opinions still occasionally arise, it is now rather generally admitted that words do not need to be present. On the other hand, several recent psychologists, even while insisting on words,³⁶ have noticed, as we do, the intervention of vague images which do not truly represent but symbolize ideas.³⁷

I shall not undertake to review these works; but I cannot resist the temptation to reproduce the highly interesting communication which has been kindly addressed to me by Professor Roman Jakobson, who, besides his well-known linguistic work, takes a fruitful interest in psychological subjects. It reads thus:

"Signs are a necessary support of thought. For socialized thought (stage of communication) and for the thought which is being socialized (stage of formulation), the most usual system of signs is language properly called; but internal thought, especially when creative, willingly uses

³⁶ See Delacroix, *Le Langage et la Pensée*, pp. 384 ff. and compare the footnote of p. 406.

³⁷ See also Titchener's *Experimental Psychology of the Thought Processes*, especially Lecture I and corresponding Notes.

other systems of signs which are more flexible, less standardized than language and leave more liberty, more dynamism to creative thought. . . . Amongst all these signs or symbols, one must distinguish between conventional signs, borrowed from social convention and, on the other hand, personal signs which, in their turn, can be subdivided into constant signs, belonging to general habits, to the individual pattern of the person considered and into episodic signs, which are established ad hoc and only participate in a single creative act.”

This remarkably precise and profound analysis beautifully enlightens our observations such as we have reported above. That there should be such an agreement between minds working in quite different branches is a remarkable fact.

Comparison with Another Question Concerning Imagery. Images constitute, as we could say, the chief subject of Taine's celebrated work *On Intelligence*. He treats them from points of view rather different from ours (as tense thought is rarely, if ever, considered). However, there is, concerning them, a question which particularly interests him and on which the above observations might possibly throw some light. As he persistently notices, it ought to be explained how images appear to us, often very vividly, and nevertheless remain distinct from real sensations; how our mind can generally differentiate between images and hallucinations.³⁸

But, in our case, we have also a sequel of images developing parallel to thought properly called. Both mental

³⁸ The same question also assumes great importance in some psychological studies of Varendonck. See his book, *Psychology of Day Dreams*, especially Chap. II, pp. 75-86.

streams, images and reasonings, constantly guide each other though keeping perfectly distinct and even, to a certain extent, independent; and we have found this to be due to a cooperation between proper consciousness and fringe-consciousness. It may be supposed that there is some analogy between the two phenomena and that one could help us to understand the other.

Can Imagery be Educated? The above considerations suggest a question analogous to one which has been raised at the end of Section IV. Is it possible—if desirable—for our volition to influence the nature of the auxiliary signs used by your thought? Now, this has been done. Titchener has carried out a most remarkable attempt in that sense. As he explains to us,³⁹ his natural tendency would have been to employ internal speech; but he has always tried and always succeeded in having a wide range and a great variety of imagery, “fearing that, as one gets older, one tends also to become more and more verbal in type.”

That too great importance of verbal intervention in his thought is thus prevented by a constant renewing of imagery. What is more curious, he uses for such a purpose not only visual images but, above all, auditory, viz., musical ones.

But he uses also the help of visual imagery, “which is always at my disposal,” he says, “and which I can mould and direct at will.” “Reading any work, I instinctively arrange the facts or arguments in some visual pattern and I am as likely to think in terms of this pattern as I am to think in words,” and the better the work suits such a pattern, the better it is understood.

Such an auto-education of mental processes seems to me

³⁹ *Experimental Psychology of the Thought Processes*, pp. 7 ff.

to be one of the most remarkable achievements in psychology.

General Remarks. All this concerns men engaged in intellectual work. Investigation among other groups seems to meet the difficulty that, as we have seen, the laws of tense thought may be and seem to be very different from those of usual and common ideation, which is the only frequent one among ordinary people. This is probably the reason why Galton, though seeing the necessity of a more general inquiry, was unable to make it.

At any rate, we see that, while what we said in the preceding sections seems to be common to various creative minds, the nature of auxiliary concrete representations may vary considerably from one mind to the other.