VIII. PARADOXICAL CASES OF INTUITION

IF, IN SOME exceptionally intuitive minds, ideas may evolve and combine in still deeper unconscious layers than in the above-mentioned cases, then even important links of the deduction may remain unknown to the thinker himself who has found them. The history of science offers some remarkable examples.

Fermat (1601-1661). Pierre de Fermat was a magistrate, a counselor at the Parliament of Toulouse. It was a time when life was less complicated than nowadays, and the requirements of his duties apparently did not hamper him in his mathematical researches, which were considerable. Besides having participated in the origins of infinitesimal calculus and even in the creation of calculus of probabilities, he dealt actively with arithmetical questions. Among the ancient mathematicians whose works were in his possession, he owned a translation of the work of Diophantes, a Greek author who had dealt with such arithmetical subjects. Now, at Fermat's death, his copy of Diophantes' work was found to bear in the margin the following observation (in Latin):

"I have proved that the relation $x^m + y^m = z^m$ is impossible in integral numbers (x, y, z different from 0; m greater than 2); but the margin does not leave me room enough to inscribe the proof."

Three centuries have elapsed since then, and that proof which Fermat could have written in the margin had the latter been a little broader, is still sought for. However, Fermat does not seem to have been mistaken, for partial proofs have been found, viz., proofs for some extended classes of values of the exponent m: for instance, the proof has been obtained for every *m* not greater than 100. But the work—an immense one—which made it possible to get these partial results could not be accomplished by direct arithmetical considerations:[1] it required the help of some important algebraic theories of which no knowledge existed in the time of Fermat *and no conception appears in his writings*. After several fundamental principles of algebra had been laid down during the eighteenth century and at the beginning of the nineteenth, the German mathematician Kummer, in order to attack that question of the "last theorem of Fermat," was obliged to introduce a new and audacious conception, the "ideals," a grand idea which entirely revolutionized algebra. As we just said, even that powerful tool given to mathematical thought allows, as yet, only a partial proof of the mysterious theorem.

Riemann (1826-1866). Bernhard Riemann, whose extraordinary intuitive power we have already mentioned, has especially renovated our knowledge of the distribution of prime numbers, also one of the most mysterious questions in mathematics.[2] He has taught us to deduce results

1 The use of considerations of that kind has been attempted by the most prominent masters—beginning with Abel—during the last two centuries. Every significant gain which can be obtained in that direction seems to have been reached, and those gains are quite limited ones. The French Academy of Sciences in Paris yearly receives several papers on that subject, most of which are absurd, while a few reproduce known results of Abel or others.

2 Both those instances concerning Fermat and Riemann relate to arithmetic. Indeed, arithmetic, which is the first study in elementary teaching, is one of the most difficult, if not the most difficult branch of mathematics, when one tries to penetrate it more deeply. Essential gains are generally obtained, as happens in our examples, on an arithmetical question by reconducting it to higher algebra or to the infinitesimal calculus.

It must be observed that the example of this discovery of Riemann in that line from considerations borrowed from the integral calculus: more precisely, from the study of a certain quantity, a function of a variable s which may assume not only real, but also imaginary values. He proved some important properties of that function, but enunciated two or three as important ones without giving the proof. At the death of Riemann, a note was found among his papers, saying "These properties of $\zeta(s)$ (the function in question) are deduced from an expression of it which, however, I did not succeed in simplifying enough to publish it."

We still have not the slightest idea of what the expression could be. As to the properties he simply enunciated, some thirty years elapsed before I was able to prove all of them but one. The question concerning that last one remains unsolved as yet, though, by an immense labor pursued throughout this last half century, some highly interesting discoveries in that direction have been achieved. It seems more and more probable, but still not at all certain, that the "Riemann hypothesis" is true. Of course, all these complements could be brought to Riemann's publication only by the help of facts which were completely unknown in his time; and, for one of the properties enunciated by him, it is hardly conceivable how he can have found it without using some of these general principles, no mention of which is made in his paper.

Galois (1811-1831). Most striking is the personality of Evariste Galois whose tragic life, abruptly ended in his

again illustrates the difference between two aspects of intuition which Poincaré believed to be identical. In general, Riemann's intuition, as Poincaré observes, is highly geometrical; but this is not the case for his memoir on prime numbers, the one in which that intuition is the most powerful and mysterious: in that memoir, there is no important role of geometrical elements.

early youth, brought to science one of the most capital monuments we know of. Galois' passionate nature was captivated by mathematical science from the moment he became acquainted with Legendre's geometry. However, he was violently dominated by another overpowering feeling, enthusiastic devotion to republican and liberal ideas, for which he fought in a passionate and sometimes very imprudent way. Nevertheless, the death he met with at the age of twenty did not occur in that struggle, but in an absurd duel.

Galois spent the night before that duel in revising his notes on his discoveries. First, the manuscript which had been rejected by the Academy of Sciences as being unintelligible (one may not wonder that such highly intuitive minds are very obscure) ; then, in a letter directed to a friend, scanty and hurried mention of other beautiful views, with the same words hastily and repeatedly inscribed in the margin "I have no time." Indeed, few hours remained to him before going where death awaited him.

All those profound ideas were at first forgotten and it was only after fifteen years that, with admiration, scientists became aware of the memoir which the Academy had rejected. It signifies a total transformation of higher algebra, projecting a full light on what had been only glimpsed thus far by the greatest mathematicians, and, at the same time, connecting that algebraic problem with others in quite different branches of science.

But what especially belongs to our subject is one point in the letter written by Galois to his friend and enunciating a theorem on the "periods" of a certain kind of integrals. Now, this theorem, which is clear for us, could not have been understood by scientists living at the time of Galois:

these "periods" had no meaning in the state of science of that day; they acquired one only by means of some principles in the theory of functions, today classical, but which were not found before something like a quarter of a century after the death of Galois. It must be admitted, therefore: (1) that Galois must have conceived these principles in some way; (2) that they must have been unconscious in his mind, since he makes no allusion to them, though they by themselves represent a significant discovery.

The case of Galois deserves some attention in connection with our former distinction. In some ways he reminds us of Hermite. He is, like him, a thoroughly analytical mathematician, though he came to his first and enthusiastic vision of science by the geometry of Legendre. One of his early essays while a schoolboy was of a geometrical nature, but it was the only one. A curious thing is that Galois' teacher in mathematics in the high school, Mr. Richard, who had the merit of discovering at once his extraordinary abilities, was also, fifteen years later, the teacher of Hermite; this, however, cannot be regarded otherwise than as a mere coincidence, since the genius of such men is evidently a gift of nature, independent of any teaching.

On the other hand, Galois, who was evidently highly intuitive according to our definition (A), does not appear as such in terms of the definition (B). In the proof of the general theorem which affords a definitive solution to the main problem of algebra, there is no trace of "scattered ideas," no combination of apparently heterogeneous principles: his thought is, so to speak, an intensive and not an extensive one; and I should be inclined to say as much of the discoveries contained in his posthumous letter (the letter written during the night just before his fatal duel), though the stream of thought cannot be so surely characterized on such a sequel of simply and briefly enunciated results. This does not exclude the occasional possibility of a connection between the aspects (A) and (B) of intuition; but in Galois' case, they appear to be independent of each other.

From the second point of view, it is clear that Galois profoundly differs from Hermite, whose discovery concerning quadratic forms is a typical example of "thinking aside."

A Case in the Work of Poincaré. It seems to have been unnoticed that something similar occurs in Poincaré's *Méthodes Nouvelles de la Mécanique Celeste.* In his Volume III (see p. 261), he has to deal with the calculus of variations and he uses a sufficient condition for a minimum, equivalent to the one which results from Weierstrass's method (see above, p. 111). But he does not give a proof of that condition: he speaks of it as a known fact. Now, as we have said, Weierstrass's method was not published at the time when that volume of the *Méthodes Nouvelles* was written. Moreover, he does not make any mention of Weierstrass's discovery, which he should have necessarily done if he had received any private communication of it. Above all, it must be added that the condition is formulated in a form slightly different (though basically equivalent) from the one which is classically known as resulting from Weierstrass's method. Must we think that Weierstrass's argument or an analogous one was found by Poincaré and remained unconscious in his mind?[3]

[3] The case appears a still stranger one if we notice that at the same page (p. 261) of his third volume, just a few lines before, Poincaré writes: "Cette recherche se rattache à la difficile question de la variation seconde."

Historical Comparisons. In such cases, we must admit that some parts of the mental process develop so deeply in the unconscious that some parts of it, even important ones, remain hidden from our conscious self. We come very near the phenomena of dual personality such as were observed by psychologists of the nineteenth century.

Even intermediaries seem to have existed between the two kinds of phenomena. I think of Socrates' ideas being suggested to him by a familiar demon or also of the nymph Egeria whom Numa Pompilius used to consult frequently.

An analogous example can possibly be spoken of in the mathematical field. It is Cardan, who is not only the inventor of a well-known joint which is an essential part of automobiles, but who has also fundamentally transformed mathematical science by the invention of imaginaries. Let us recall what an imaginary quantity is. The rules of algebra show that the square of any number, whether positive or negative, is a positive number: therefore, to speak of the square root of a negative number is mere absurdity. Now, Cardan deliberately commits that absurdity and begins to calculate on such "imaginary" quantities.

One would describe this as pure madness; and yet the whole development of algebra and analysis would have been

Now, in Weierstrass's theory—that is, in our present view of the calculus of variations—there is no question of the second variation, which is fully left out of consideration.

Therefore, there is a curious contradiction in that page of the *Méthodes Nouvelles.* The allusion to the "variation seconde" is that of a man having no idea of the new theory. On the contrary, Poincaré proves to have been fully aware of it when he enunciated his condition (A) (his form of Weierstrass's condition): nobody thought of anything of that kind in the older ideas on the subject; one only knew of the classic (but less adequate) "Legendre condition."

Should one think of the case of Poincaré as of a kind of dual personality?

impossible without that fundament—which, of course, was, in the nineteenth century, established on solid and rigorous bases. It has been written that the shortest and best way between two truths of the real domain often passes through the imaginary one.

We have mentioned Cardan's case with Socrates' and Numa Pompilius', because he too is reported by some of his biographers to have received suggestions from a mysterious voice at certain periods of his life. However, testimonies on that point do not agree at least in details.