one or more of the summands are restricted to be small (so that the corresponding weight $x^{\frac{r}{k}-1}$ is large).

The proof of (1.43) is a standard but lengthy application of the Hardy–Littlewood circle method, and is beyond the scope of this book. The reader may consult [379] for the full proof. □

Wooley [382] shown that one can set $k_0 = O(r \log r)$. This is (up to a constant factor) also the best current bound for k in (1.42). His proof also relies on Theorem 1.37, but the number-theoretic part is different.

Exercise

1.9.1 In the proof of Proposition 1.50, verify that with probability 1 one has $E(n) = O_{C,k,r,B}(1)$ for all but finitely many n.

1.10 Appendix: the distribution of the primes

Several results in this chapter relied on facts concerning the distribution of the primes

$$P = \{2, 3, 5, \ldots\}.$$

The distribution of this set is of course a very well-studied subject in analytic number theory, with one of the fundamental results being the *prime number theorem*

$$|P \cap [1, n]| = (1 + o(1))\frac{n}{\log n}. \tag{1.44}$$

An equivalent formulation is that if p_k denotes the kth prime, then $p_k = (1 + o(1))k \log k$. The famous *Riemann hypothesis*, which is still unsolved, is equivalent to the stronger statement that

$$|P \cap [1, n]| = \int_2^n \frac{dx}{\log x} + O_\varepsilon\left(n^{1/2+\varepsilon}\right) \tag{1.45}$$

for any $\varepsilon > 0$, or equivalently that $p_k = k \log k + O_\varepsilon(k^{1/2+\varepsilon})$ for any $\varepsilon > 0$.

The prime number theorem is rather deep and will not be proven here. In this Appendix we present some related results, most of which have surprisingly elementary and beautiful proofs. As they are number-theoretical rather than probabilistic in nature we have chosen to place these results in an appendix to this chapter.

We begin with some classical estimates of Chebyshev and Mertens). As is customary, when summing over a variable p, p is understood to denote a prime.

Proposition 1.51 (Elementary prime number estimates) *Let $n \geq 1$ be an integer. Then we have the estimates*

$$\sum_{p \leq n} \log p = O(n) \tag{1.46}$$

$$\sum_{p \leq n} \frac{\log p}{p} = \log n + O(1) \tag{1.47}$$

$$\sum_{p \leq n} \frac{1}{p} = \log \log n + O(1). \tag{1.48}$$

Remark 1.52 With the prime number theorem, we can improve (1.46) to $\sum_{p \leq n} \log p = (1 + o(1))n$, but it is not necessary to do so for our applications here.

Proof We first prove (1.46). Without loss of generality we may take n to be a power of two. Consider the binomial $\binom{2n}{n}$. From Pascal's formula we know that $\binom{2n}{n} \leq 4^n$. On the other hand, it is clear that every prime between n and $2n$ will divide $\binom{2n}{n}$. Thus

$$\prod_{n < p \leq 2n} p \leq 4^n.$$

Taking logarithms we conclude

$$\sum_{n < p \leq 2n} \log p = O(n).$$

Applying this bound to $n/2$, $n/4$, and so forth, and then summing the geometric series, the claim (1.46) follows.

Now we prove (1.47). This is a similar argument but based around the factorial $n!$ instead of $\binom{2n}{n}$. Observe that the only primes dividing $n!$ are those less than or equal to n. For each prime $p \leq n$, there are $\lfloor n/p \rfloor$ numbers (between 1 and n) divisible by p, $\lfloor n/p^2 \rfloor$ numbers (between 1 and n) divisible by p^2 and so on. Thus

$$n! = \prod_{p \leq n} p^{\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \cdots}. \tag{1.49}$$

Taking the logarithm of both sides and applying Stirling's formula (Exercise 1.10.1) we obtain

$$n \log n + O(n) = \sum_{p \leq n} (\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \cdots) \log p.$$

Since

$$\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \cdots = \frac{n}{p} + O(1) + O\left(\frac{n}{p^2}\right),$$

we conclude, after some rearranging, that

$$\sum_{p \leq n} n \frac{\log p}{p} = n \log n + O(n) + \sum_{p \leq n} O(\log p) + \sum_{p \leq n} O\left(\frac{n \log p}{p^2}\right).$$

Since $\sum_k \frac{\log k}{k^2}$ is convergent, the last term is $O(n)$. The claim now follows from (1.46).

We shall deduce (1.48) from (1.47) using Abel's summation technique, rewriting one partial sum over primes as an average of others. Observe from the fundamental theorem of calculus that

$$\begin{aligned} \frac{1}{p} &= \frac{\log p}{p} \frac{1}{\log p} \\ &= \frac{\log p}{p} \int_1^\infty \mathbf{I}(t > p) \frac{dt}{t \log^2 t} \end{aligned}$$

and hence

$$\sum_{p \leq n} \frac{1}{p} = \sum_{p \leq n} \frac{\log p}{p} \int_1^\infty \mathbf{I}(t > p) \frac{dt}{t \log^2 t}.$$

Swapping the sum and integral, we obtain

$$\sum_{p \leq n} \frac{1}{p} = \int_1^\infty \left(\sum_{p \leq t} \frac{\log p}{p} \right) \frac{dt}{t \log^2 t}.$$

Applying (1.47), we obtain

$$\sum_{p \leq n} \frac{1}{p} = \int_1^\infty (\log t + O(1)) \frac{dt}{t \log^2 t}.$$

Since $\log t \frac{t}{\log^2 t}$ is the antiderivative of $\log \log t$, and $\frac{1}{t \log^2 t}$ is absolutely convergent, the claim follows. □

We now turn to a deeper fact concerning the distribution of primes in intervals.

Theorem 1.53 *For all sufficiently large n, we have $|P \cap [n - x, n)| = \Theta(\frac{x}{\log n})$ for all $n^{2/3} < x < n$.*

Results of this type first appeared by Hoheisel [183]; the result as claimed is due to Ingham [188]. Note that this theorem follows immediately from the Riemann hypothesis (1.45). However, this theorem can be proven without using the Riemann hypothesis, rather some weaker (but still very non-trivial) facts on the distribution of zeroes of the Riemann zeta function: see [170]. We remark that if one only seeks the upper bound on $|P \cap [n - x, n)|$ then one can use relatively elementary sieve theory methods to establish the claim. The constant $2/3$ has been lowered (the current record is $7/12$, see [187], [178]). However, for the applications here, any exponent less than 1 will suffice.

We now combine this theorem with the Abel summation method to establish some further estimates on sums involving primes.

Proposition 1.54 *Let n be a large integer. Then we have the estimates*

$$\sum_{p\in P\cap[1,n-n^{2/3})} \frac{1}{n-p} = \Theta(1) \tag{1.50}$$

$$\sum_{p\in P\cap[1,n-n^{2/3})} \frac{\log(n-p)}{n-p} = \Theta(\log n). \tag{1.51}$$

Proof We begin by proving (1.50). From the fundamental theorem of calculus we have

$$\frac{1}{n-p} = \int_1^\infty 1_{p\in[n-x,n-n^{2/3})} \frac{1}{x^2}\,dx$$

for all $p \in P \cap [1, n - n^{2/3})$, and hence

$$\sum_{p\in P\cap[1,n-n^{2/3})} \frac{1}{n-p} = \int_1^\infty \left|P \cap \left[n-x, n-n^{2/3}\right)\right| \frac{dx}{x^2}.$$

The integrand vanishes when $x \leq n^{2/3}$. When $n^{2/3} < x \leq 2n^{2/3}$, Theorem 1.53 shows that the integrand is $O(\frac{1}{n^{2/3}\log n})$, while for $x \geq n^{2/3}$ another application of Theorem 1.53 shows that the integrand is $\Theta(\frac{1}{x\log n})$ when $x \leq n$ and $\Theta(\frac{n}{x^2\log n})$ when $x > n$. Putting all these estimates together we obtain (1.50). The estimate (1.51) then follows immediately from (1.50) since $\log(n-p) = \Theta(\log n)$ when $p \in [1, n - n^{2/3}]$. □

Exercises

1.10.1 By approximating the sum $\sum_{m=1}^n \log m$ by the integral $\int_1^n \log x\,dx$, prove *Stirling's formula*

$$\log n! = n\log n - n + O(\log n) \tag{1.52}$$

for all $n > 1$.

1.10.2 Using Proposition 1.51, show that there is a constant c so that there is always a prime between n and cn for every positive integer n.

1.10.3 By being more careful in the proof of (1.46), show that

$$\sum_{p<n} \log p \leq 2n\log 2 + O\left(n^{1/2}\right)$$

and

$$\sum_{n \le p < 2n} \log p + \sum_{p \le 2n/3} \log p \ge 2n \log 2 - O\big(n^{1/2}\big),$$

and conclude *Bertrand's postulate*, namely that for every sufficiently large integer n there exists a prime between n and $2n$. (This argument is due to Ramanujan. Bertrand's postulate in fact holds for all integers n, as the case of small n can be verified directly.)

1.10.4 Without using the prime number theorem, prove that $|P \cap [1, n]| = \Theta(\frac{n}{\log n})$; this is known as *Chebyshev's theorem*. This theorem is of course superseded by the prime number theorem $\pi(n) = (1 + o(1))\frac{n}{\log n}$, but has the advantage of having a short elementary proof.

1.10.5 Prove that $p_k = \Theta(k \log k)$, where p_k denotes the kth prime. Again, this is superseded by the prime number theorem $p_k = (1 + o(1))k \log k$.

1.10.6 Define the *von Mangoldt function* $\Lambda : \mathbf{Z}^+ \to \mathbf{R}$ by setting $\Lambda(n) := \log p$ if $n > 1$ is a power of a prime p, and $\Lambda(n) = 0$ otherwise. Show that

$$\sum_{d|n} \Lambda(d) = \log n \tag{1.53}$$

for all integers $n \ge 1$. Use this to prove that

$$\left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) = \sum_{n=1}^{\infty} \frac{\log n}{n^s}$$

for all real numbers $s > 1$. Also, use (1.53) to give an alternative proof of (1.49).

1.10.7 Using the preceding exercise, show that

$$\sum_{n=1}^{\infty} \frac{\log p}{p^s} = \frac{1}{s-1} + O(1)$$

for all $s > 1$; integrate this to conclude

$$\sum_{n=1}^{\infty} \frac{1}{p^s} = \log \frac{1}{s-1} + O(1) \tag{1.54}$$

for all $s > 1$. Show that these estimates can also be deduced from Proposition 1.51 via Abel's method. Conversely, use (1.54) and (1.46) to give an alternative proof of (1.48).

1.10.8 Using Abel's summation method, show that the prime number theorem $\pi(x) = (1 + o(1))\frac{x}{\log x}$ is equivalent to the estimate $\sum_{n \le x} \Lambda(n) = (1 + o(1))x$.

1.10.9 By being more careful in the proof of (1.48), show that

$$\sum_{p<n} \frac{1}{p} = \log\log n + C + O\left(\frac{1}{\log n}\right)$$

for some absolute constant C. Use this to deduce *Merten's theorem*

$$\prod_{p<n}\left(1 - \frac{1}{p}\right) = (1 + o(1))\frac{C'}{\log n} \tag{1.55}$$

for some other absolute constant C' and all $n > 1$. (In fact one has $C' = e^{-\gamma}$, where $\gamma = 0.577\ldots$ is Euler's constant.)