and a $y \in Z$ such that

$$1 - \frac{|A \cap (B + y)|}{|Z|} \geq \left(1 - \frac{|A|}{|Z|}\right)\left(1 - \frac{|B|}{|Z|}\right).$$

1.1.6 Consider a set A as above. Show that there exists a subset $\{v_1, \ldots, v_d\}$ of Z with $d = O(\log \frac{|Z|}{|A|})$ such that

$$|A + [0, 1]^d \cdot (v_1, \ldots, v_d)| \geq |Z|/2.$$

1.1.7 Consider a set A as above. Show that there exists a subset $\{v_1, \ldots, v_d\}$ of Z with $d := O(\log \frac{|Z|}{|A|} + \log \log(10 + |Z|))$ such that

$$A + [0, 1]^d \cdot (v_1, \ldots, v_d) = Z.$$

1.2 The second moment method

The first moment method allows one to control the order of magnitude of a random variable X by its expectation $\mathbf{E}(X)$. In many cases, this control is insufficient, and one also needs to establish that X usually does not deviate too greatly from its expected value. These types of estimates are known as *large deviation inequalities*, and are a fundamental set of tools in the subject. They can be significantly more powerful than the first moment method, but often require some assumptions concerning independence or approximate independence.

The simplest such large deviation inequality is *Chebyshev's inequality*, which controls the deviation in terms of the variance $\mathbf{Var}(X)$:

Theorem 1.5 (Chebyshev's inequality) *Let X be a random variable. Then for any positive λ*

$$\mathbf{P}\big(|X - \mathbf{E}(X)| > \lambda \mathbf{Var}(X)^{1/2}\big) \leq \frac{1}{\lambda^2}. \tag{1.8}$$

Proof We may assume $\mathbf{Var}(X) > 0$ as the case $\mathbf{Var}(X) = 0$ is trivial. From Markov's inequality we have

$$\mathbf{P}(|X - \mathbf{E}(X)|^2 > \lambda^2 \mathbf{Var}(X)) \leq \frac{\mathbf{E}(|X - \mathbf{E}(X)|^2)}{\lambda^2 \mathbf{Var}(X)} = \frac{1}{\lambda^2}$$

and the claim follows. □

Thus Chebyshev's inequality asserts that $X = \mathbf{E}(X) + O(\mathbf{Var}(X)^{1/2})$ with high probability, while in the converse direction it is clear that $|X - \mathbf{E}(X)| \geq \mathbf{Var}(X)^{1/2}$ with positive probability. The application of these facts is referred to as the *second moment method*. Note that Chebyshev's inequality provides both upper tail and lower tail bounds on X, with the tail decaying like $1/\lambda^2$ rather than $1/\lambda$. Thus