1.3.12 With the hypotheses of Corollary 1.9, establish the inequality

$$\mathbf{E}\left(\binom{X}{n}\right) \leq \frac{1}{n!}\mathbf{E}(X)^n$$

for all $n \in \mathbf{N}$. (Hint: expand $\binom{X}{n}$ as $\sum_{i_1<\cdots<i_n} t_{i_1}\cdots t_{i_n}$). Use this (and Stirling's formula (1.52)) to derive an inequality similar to that in Corollary 1.9 in the case $\epsilon > 1$. For a generalization of this inequality, see Lemma 1.40 below.

1.4 Correlation inequalities

Chernoff's inequality is useful for controlling quantities of the form $t_1 + \cdots + t_n$ where $t_1, \ldots, t_n$ are independent variables. In many applications, however, one needs to instead control more complicated polynomial expressions of $t_1, \ldots, t_n$, such as monotone quantities.

Definition 1.17 (Monotone increasing variables) Let $t_1, \ldots, t_n$ be jointly independent boolean random variables. A random variable $X = X(t_1, \ldots, t_n)$ is *monotone increasing* if we have

$$X(t_1, \ldots, t_n) \geq X(t_1', \ldots, t_n') \text{ whenever } t_i \geq t_i' \text{ for all } 1 \leq i \leq n$$

or equivalently if X is monotone increasing in each of the variables t_i separately. We call X *monotone decreasing* if $-X$ is monotone increasing. We say that an event A is *monotone increasing* (resp. decreasing) if the indicator $\mathbf{I}(A)$ is monotone increasing (resp. decreasing).

Example 1.18 If $P(t_1, \ldots, t_n)$ is any polynomial of $t_1, \ldots, t_n$ with non-negative coefficients, then P is monotone increasing and $-P$ is monotone decreasing, and the event $P(t_1, \ldots, t_n) \geq k$ is monotone increasing for any fixed k.

It is reasonable to think that any two increasing (resp. decreasing) variables or events are, in some way, positively correlated; intuitively, if both X and Y are monotone increasing (resp. decreasing), then the event that X is large (resp. small) should boost up the chance that Y is also large (resp. small). This intuition was materialized by Fortuin, Kasteleyn and Ginibre [104], motivated by problems in statistical mechanics:

Theorem 1.19 (FKG inequality) *Let $n \geq 0$, and let X and Y be two monotone increasing variables. Then*

$$\mathbf{E}(XY) \geq \mathbf{E}(X)\mathbf{E}(Y)$$

or equivalently

$$\mathbf{Cov}(X, Y) \geq 0.$$

The same inequality holds for the case both X and Y are monotone decreasing.

Proof By replacing X, Y with $-X, -Y$ if necessary, we may assume that X and Y are both monotone increasing.

We use induction on n. The base case $n = 0$ is trivial since in this case X and Y are deterministic. Now assume inductively that $n \geq 1$ and the claim has already been proven for $n - 1$. We may assume that $\mathbf{P}(t_n = 0)$ and $\mathbf{P}(t_n = 1)$ are non-zero since otherwise the claim follows immediately from the induction hypothesis. Observe that the covariance $\mathbf{Cov}(X, Y)$ is unaffected if we shift X and Y by constants. Thus we may normalize

$$\mathbf{E}(X|t_n = 0) = \mathbf{E}(Y|t_n = 0) = 0 \tag{1.25}$$

where $\mathbf{E}(X|t_n = 0)$ denotes the conditional expectation of X relative to the event $t_n = 0$. By monotonicity of X, Y in the t_n variable and the joint independence of the t_i we then have

$$\mathbf{E}(X|t_n = 1), \mathbf{E}(Y|t_n = 1) \geq 0. \tag{1.26}$$

Observe that, conditioning on the event $t_n = 0$, the random variables X, Y are monotone increasing functions of $t_1, \ldots, t_{n-1}$. Thus by the induction hypothesis

$$\mathbf{E}(XY|t_n = 0) \geq \mathbf{E}(X|t_n = 0)\mathbf{E}(Y|t_n = 0) = 0$$

and similarly

$$\mathbf{E}(XY|t_n = 1) \geq \mathbf{E}(X|t_n = 1)\mathbf{E}(Y|t_n = 1).$$

By Bayes' formula we thus have

$$\begin{aligned}\mathbf{E}(XY) &= \mathbf{E}(XY|t_n = 0)\mathbf{P}(t_n = 0) + \mathbf{E}(XY|t_n = 1)\mathbf{P}(t_n = 1) \\ &\geq \mathbf{E}(X|t_n = 1)\mathbf{E}(Y|t_n = 1)\mathbf{P}(t_n = 1).\end{aligned}$$

On the other hand, from (1.25) and another application of the total probability formula we have

$$\mathbf{E}(X)\mathbf{E}(Y) = \mathbf{E}(X|t_n = 1)\mathbf{P}(t_n = 1)\mathbf{E}(Y|t_n = 1)\mathbf{P}(t_n = 1).$$

Since $\mathbf{P}(t_n = 1) \leq 1$, the claim now follows from (1.26). □

From (1.1) and an easy induction we have an immediate corollary to Theorem 1.19:

Corollary 1.20 *Let A and B be two increasing events, then*

$$\mathbf{P}(A \wedge B) \geq \mathbf{P}(A)\mathbf{P}(B).$$