If we choose K sufficiently large depending on ε, we thus see from Markov's inequality that

$$\mathbf{P}(j \text{ is good}) \geq \frac{1}{2}.$$

Now we come to the final and most important observation: For any fixed j, the event that x_j is good is a monotone increasing random variable, with respect to indicator variables $t_l := \mathbf{I}(l \in B)$. Thus, by Corollary 1.20,

$$\mathbf{P}(j \text{ is good for all } j \in [0, J]) \geq \prod_{j \in [0,J]} \mathbf{P}(j \text{ is good}) \geq 2^{-J-1}.$$

Since $J = \frac{1}{4} \ln_2 n + O(1)$ and n is assumed to be large, the claim (1.29) follows. □

Exercises

1.4.1 Deduce Theorem 1.22, from Lemma 1.24. (Hint: the convergence of the geometric series $1 + g + g^2 + \cdots$ for $|g| < 1$ may be useful at one point.)

1.4.2 Deduce Corollary 1.23, from Theorem 1.22.

1.4.3 Let the notation and assumptions be as in Theorem 1.19. Suppose that each of the independent variables $t_1, \ldots, t_n$ attain the values 0 and 1 with positive probability. Show that equality holds in Theorem 1.19 if and only if X and Y depend on disjoint subsets of the random variables $t_1, \ldots, t_n$.

1.5 The Lovász local lemma

Let $(A_i)_{i \in V}$ be a finite collection of events in a probabilistic space; we will later view the index set V as the vertex set of a graph. In many situations, it is desirable to show that there is a chance that the complementary events $(\bar{A}_i)_{i \in V}$ hold simultaneously, i.e. that $\mathbf{P}(\bigwedge_{i \in V} \bar{A}_i) > 0$. This is particularly useful when the A_i are bad events that we would like to avoid.

If the A_i are mutually independent, then the problem is trivial, as we have

$$\mathbf{P}\left(\bigwedge_{i \in V} \bar{A}_i\right) = \prod_{i \in V} \mathbf{P}(\bar{A}_i) = \prod_{v \in V}(1 - \mathbf{P}(A_i)), \tag{1.30}$$

which is positive if $\mathbf{P}(A_i)$ are all strictly less than one. On the other hand, mutual independence is a very strong assumption which rarely holds.

One may expect that something similar to (1.30) is still true if we allow a sufficiently "local" dependence among the A_is, so that we still have good control on $\mathbf{P}(A_i)$ even after conditioning on most of the events $\bar{A}_j$. This is indeed possible, as shown by Lovász in 1975 in a joint paper with Erdős [93]. We present a modern version of this lemma as follows.

Lemma 1.25 (Lovász local lemma) *Let V be a finite set, and for each $i \in V$ let A_i be a probabilistic event. Assume that there is a directed graph $G(V, E)$ (without loops) on the vertex set V (which is known as the* dependency graph *of the A_i); and a sequence of numbers $0 \leq x_i < 1$ for each $i \in V$ such that the estimate*

$$\mathbf{P}\left(A_i | \bigwedge_{j \in S} \bar{A}_j\right) \leq x_i \prod_{(i,j) \in E} (1 - x_j) \tag{1.31}$$

holds whenever $i \in V$; and $S \subseteq V \backslash \{i\}$ is such that $\bigwedge_{j \in S} \bar{A}_j$ has non-zero probability and $(i, j) \notin E$ for all $j \in S$. Then for any disjoint $S, S' \subseteq V$ we have

$$\mathbf{P}\left(\bigwedge_{i \in S} \bar{A}_i | \bigwedge_{i \in S'} \bar{A}_i\right) \geq \prod_{i \in S} (1 - x_i) > 0. \tag{1.32}$$

In particular we have

$$\mathbf{P}\left(\bigwedge_{i \in V} \bar{A}_i\right) \geq \prod_{i \in V} (1 - x_i) > 0.$$

The graph G is usually referred to as the *dependency* graph of the A_i. Note that (1.31) will hold if we have

$$\mathbf{P}(A_i) \leq x_i \prod_{(i,j) \in E} (1 - x_j)$$

and each A_i is mutually independent to all of the A_j with $(i, j) \notin E$ and $j \neq i$. This was in fact the hypothesis stated in the original formulation of the lemma. However, there are situations where these rather strong mutual independence hypotheses are not available and one needs the full strength of Lemma 1.25. Alon and Spencer's book [12] Chapter 5 contains many interesting applications.

Proof of Lemma 1.25 We shall induce on the total cardinality $|S| + |S'|$. If $|S| + |S'| = 0$ then S, S' are empty, and the claim (1.32) is trivial. Now assume inductively that $|S| + |S'| \geq 1$, and the claim has already been proven for smaller values of $|S| + |S'|$. Note that the case $|S| = 0$ is trivial. To establish the claim for $|S| \geq 1$, it suffices to do so for the case $|S| = 1$. Indeed, if $|S| \geq 1$, then we can split $S = \{j\} \cup (S \backslash \{j\})$ for some $j \in S$. From the definition of conditional probability we have

$$\mathbf{P}\left(\bigwedge_{i \in S} \bar{A}_i | \bigwedge_{i \in S'} \bar{A}_i\right) = \mathbf{P}\left(\bar{A}_i | \bigwedge_{i \in S' \cup S \backslash \{j\}} \bar{A}_i\right) \mathbf{P}\left(\bigwedge_{i \in S \backslash \{j\}} \bar{A}_i | \bigwedge_{i \in S'} \bar{A}_i\right)$$

and the claim (1.32) then follows by applying the induction hypothesis to estimate the second factor.