Set $f := 1_A/\tau$. Clearly f is non-negative and $f \leq \nu$. Also $\mathbf{E}_Z(f) \geq \delta \mathbf{P}_Z(B) = \Theta(\delta\tau)$. From Lemma 10.22 and (10.11) we have

$$|\mathrm{Spec}_\alpha(f)| \leq 4/\alpha^2 \text{ whenever } \alpha = o\big(|Z|^{-1/10}\big),$$

while from (4.2) we have the very crude bound $\|\hat{f}\|^2_{l^2(Z)} \leq \tau^{-2} \leq |Z|^{0.02}$. Combining these two estimates, we easily obtain

$$\|\hat{f}\|_{l^{5/2}(Z)} = O(1) \tag{10.12}$$

(for instance); see Exercise 10.2.2. Applying Theorem 10.20 (with $\eta := |Z|^{-1/5}$), we conclude

$$\Lambda_3(f, f, f) \geq 8p^{-12/\delta} - O_p\big(\log^{-1/5}|Z|\big).$$

On the other hand, if A contained no arithmetic progressions of length 3, then we would have

$$\Lambda_3(f, f, f) = \frac{|A|}{\tau^3|Z|^2} = O\left(\frac{1}{\tau^3|Z|}\right) = O(|Z|^{-0.97}),$$

which would lead to a contradiction if Z was large compared with δ, p, and the claim follows. □

We remark that the above argument is quite quantitative, and it is not difficult to use it to extract specific bounds for Theorem 10.18, but we will not do so here.

Exercises

10.2.1 Show that if A, B are two additive sets (possibly in different ambient groups) then $r_k(A \times B) \geq r_k(A)r_k(B)$. Conclude in particular that $r_3(F_3^n) \geq 2^n$ for all n.

10.2.2 Deduce (10.12) from the bounds on $\mathrm{Spec}_\alpha(f)$ and $\|\hat{f}\|_{l^2(Z)}$. (Hint: one can use an analog of (1.7).)

10.2.3 Show that Theorem 10.18 fails if $\tau = |Z|^{-1/2-\varepsilon}$ for any absolute constant ε. (Hint: count the number of proper progressions of length 3 in B, and remove them to create A.)

10.2.4 [248] Let $s(n, d)$ be the quantity defined in Section 9.6. Show that $s(3, d) = \Theta(r_3(F_3^n))$. In particular, we have $s(3, d) = O(3^d/d)$ for large d.

10.3 The integer case

We now sketch the proof of Roth's theorem for integers (which was the original setting for Roth's argument). We shall be somewhat brief here as the result will be superseded by the Roth–Bourgain theorem, Theorem 10.30.

As in the proof of Theorem 10.12, we need two ingredients; first, we need to show that lack of progressions in $[1, N]$ implies some linear bias, and second we need to convert this linear bias to a density increment on a sub-progression of $[1, N]$. Because $[1, N]$ is not quite a group, we cannot apply Corollary 10.10 directly. However we have the following substitute.

Proposition 10.24 (Lack of progressions implies non-uniformity) *[287] Let P be an arithmetic progression of integers, and let $A \subset P$ be such that $|A| = \delta|P|$ for some $0 < \delta \leq 1$. Assume also that $|P| \geq 100/\delta^2$, and that A contains no arithmetic progressions of length 3. Then there exists $\xi \in \mathbf{R}/\mathbf{Z}$ such that*

$$|\mathbf{E}_{n\in P}(1_A(n) - \delta)e(n\xi)| = \Omega(\delta^2).$$

Proof By a rescaling argument one can take $P = [1, N]$. By Bertrand's postulate (Exercise 1.10.3) we can find a prime p between $2N$ and $4N$. We identify A with a subset of $\mathbf{Z}_p$ in the usual manner (and give $\mathbf{Z}_p$ the standard bilinear form), and observe from (10.2) and the hypothesis on A that

$$\Lambda_3(1_A, 1_A, 1_A) = \frac{1}{p^2}|A| \leq \frac{\delta}{4N}.$$

Let us now split $1_A = f_U + f_{U^\perp}$, where $f_{U^\perp} := \delta 1_{[1,N]}$ and $f_U := 1_A - f_{U^\perp}$. A simple computation shows that

$$\Lambda_3(f_{U^\perp}, f_{U^\perp}, f_{U^\perp}) \geq \frac{\delta^3}{100}$$

(say). By hypothesis on N, we conclude

$$|\Lambda_3(f_U + f_{U^\perp}, f_U + f_{U^\perp}, f_U + f_{U^\perp}) - \Lambda_3(f_{U^\perp}, f_{U^\perp}, f_{U^\perp})| = \Omega(\delta^3).$$

The left-hand side can be split as the sum of seven terms, so at least one of them is $\Omega(\delta^3)$. For sake of discussion let us suppose that

$$|\Lambda_3(f_U, f_U, f_U)| = \Omega(\delta^3);$$

the other six cases are similar (the point being that all of them involve at least one copy of f_U). Using (10.6) and the triangle inequality, we conclude that

$$\sum_{\xi\in\mathbf{Z}_p} |\hat{f}_U(\xi)|^2 |\hat{f}_U(-2\xi)| = \Omega(\delta^3).$$

On the other hand, from Plancherel's theorem we have

$$\sum_{\xi\in\mathbf{Z}_p} |\hat{f}_U(\xi)|^2 = \|f_U\|^2_{L^2(Z)} = O\big(\|1_A\|^2_{L^2(\mathbf{Z}_p)} + \|\delta 1_{[1,N]}\|^2_{L^2(\mathbf{Z}_p)}\big) = O(\delta).$$

We thus conclude that there exists $\xi \in \mathbf{Z}_p$ such that

$$|\hat{f}_U(-2\xi)| = \Omega(\delta^2),$$

thus

$$|\mathbf{E}_{n\in[1,N]}(1_A(n) - \delta)e(2n\xi/p)| = \Omega(\delta^2).$$

The claim follows. □

Similarly, we have the following analog of Lemma 10.15.

Lemma 10.25 (Non-uniformity implies density increment) *[287] Let $f : \mathbf{Z} \to \mathbf{R}$ be a function supported on an arithmetic progression P such that $|f(n)| \leq 1$ for all n, $\sum_n f(n) = 0$, and*

$$|\mathbf{E}_{n\in P} f(n)e(n\xi)| \geq \sigma$$

for some $\xi \in \mathbf{R}/\mathbf{Z}$ and $\sigma > 0$. Then there exists a proper arithmetic progression $P' \subset P$ with $|P'| = \Omega(\sigma^2|P|^{1/2})$ and.

$$|\mathbf{E}_{n\in P'} f(n)| \geq \sigma/4$$

Proof Again we may take $P = [1, N]$. Using the Kronecker approximation theorem (Corollary 3.25) we can find an integer $1 \leq r \leq N^{1/2}$ such that $\|r\xi\|_{\mathbf{R}/\mathbf{Z}} \leq N^{-1/2}$. Let P_0 denote the progression $[1, \sigma N^{1/2}/100] \cdot r$. Then we have

$$\left|\sum_n \mathbf{E}_{x\in P_0} f(n+x)e(n\xi)e(x\xi)\right| = \left|\sum_n f(n)e(n\xi)\right| \geq \sigma N,$$

where e is defined in equation (4.1). On the other hand $x \in P_0$, we see from (4.24) that $|e(x\xi) - 1| \leq \sigma/10$, and so

$$\left|\sum_n \mathbf{E}_{x\in P_0} f(n+x)e(n\xi)(e(x\xi) - 1)\right| \leq \sum_{n\in[-N,N]} \sigma/10 \leq \sigma N/2$$

(say), and so by the triangle inequality

$$\left|\sum_n \mathbf{E}_{x\in P_0} f(n+x)e(n\xi)\right| \geq \sigma N/2.$$

In particular there exists a phase $\theta \in \mathbf{R}/\mathbf{Z}$ such that

$$\mathrm{Re} \sum_n \mathbf{E}_{x\in P_0} f(n+x)e(n\xi + \theta) \geq \sigma N/2.$$

Since f sums to zero, we have $\sum_n \mathbf{E}_{x\in P_0} f(n+x) = 0$, and hence

$$\sum_n \mathbf{E}_{x\in P_0} f(n+x)\mathrm{Re}(1 + e(n\xi + \theta)) \geq \sigma N/2.$$

Note that the sum is only non-zero when $n \in (-N, N]$. By the pigeonhole principle, there thus exists an n such that

$$\mathbf{E}_{x\in n+P_0} f(x) = \mathbf{E}_{x\in P_0} f(n+x) \geq \sigma N/4.$$