Since f is bounded and supported in $[1, N]$, we conclude in particular that

$$|(n + P_0) \cap [1, N]| \geq \sigma |P_0|/4 = \Omega\big(\sigma^2 N^{1/2}\big).$$

The claim then follows by taking $P = (n + P_0) \cap [1, N]$. □

Combining this with the preceding proposition, we conclude

Corollary 10.26 (Lack of progressions implies density increment) *Let $A \subset P$ be such that $|A| = \delta |P|$ for some $0 < \delta \leq 1$. Assume that $|P| \geq 100/\delta^2$. Suppose also that A contains no arithmetic progressions of length 3. Then there exists a proper arithmetic progression P' in P with $|P'| = \Omega(\delta^4 |P|^{1/2})$ such that we have the density increment*

$$\mathbf{P}_{P'}(A) \geq \mathbf{P}_P(A) + \Omega(\delta^2).$$

By iterating this Corollary, one can eventually show that $r_3([1, N]) = O(\frac{N}{\log\log N})$; we leave this as an exercise to the reader.

There has been some recent progress in understanding the structure of subsets of $\mathbf{Z}/N\mathbf{Z}$ which attain the minimal number of progressions of length 3 among all sets with a given density; see [65]. It may be that this will lead to an alternative proof of Roth's theorem.

Exercises

10.3.1 [287] By iterating Corollary 10.26, establish the bound $r_3(P) = O(\frac{N}{\log\log N})$ for any arithmetic progression P of integers of length N, and hence $r_3(\mathbf{Z}_N) = O(\frac{N}{\log\log N})$.

10.3.2 [372] Let $f : \mathbf{Z}_N \to \mathbf{R}^+$ be such that $0 \leq f(x) \leq 1$ for all $x \in \mathbf{Z}_N$. By using the previous exercise and arguing as in Proposition 10.17, show that

$$\Lambda_3(f, f, f) = \Omega(\exp(-\exp(O(1/\mathbf{E}_Z(f))))).$$

10.4 Quantitative bounds

In the preceding section we obtained a bound of $O(N/\log\log N)$ for the quantity $r_3([1, N])$. The main reason for this double logarithm lies in the use of Kronecker's theorem in Lemma 10.25, which reduces the size of the progression P by roughly a square root, while only increasing the density by a small amount $O(\delta^2)$. This step is so inefficient that it is worthwhile to make the other parts of the argument more complicated in order to reduce the number of times one invokes Kronecker's theorem. One such approach, due to Heath-Brown and Szemerédi, is to apply

Kronecker's theorem to a large batch of frequencies at once, rather than one at a time. It yields the following improvement[1]:

Theorem 10.27 *[177, 344] For all large N, we have $r_3(\mathbf{Z}_N), r_3([1, N]) = O(N/\log^c N)$ for some absolute constant $c > 0$.*

Proof It suffices to verify the claim for $r_3([1, N])$. We refine the arguments in the preceding section, again skipping some details. First we need the following variant of Proposition 10.24.

Proposition 10.28 (Lack of progressions implies non-uniformity) *Let $A \subset [1, N]$ be such that $|A| = \delta N$ for some $0 < \delta \leq 1$ and such that A has no proper arithmetic progressions of length 3. Suppose also that $N \geq 100/\delta^2$. Let p be a prime between N and $2N$, and identify $[1, N]$ with a subset of $\mathbf{Z}_p$. Let $f_U : \mathbf{Z}_p \to \mathbf{R}$ be the function $f_U := 1_A - \delta 1_{[1,N]}$. Then there exists a set $S \subset \mathbf{Z}_N$ such that $|S| = O(\delta^{-3})$ and*

$$\sum_{\xi \in S} |\hat{f}_U(\xi)|^2 = \Omega\big(\delta^2 |S|^{1/5}\big).$$

Proof Write $f_{U^\perp} := \delta 1_{[1,N]}$. Arguing as in Proposition 10.28, we conclude once again that

$$\sum_{\xi \in \mathbf{Z}_p} |\hat{f}_{U^\perp}(\xi)|^2 |\hat{f}_U(-2\xi)| = \Omega(\delta^3)$$

or something very similar to this. A direct calculation (which we leave as an exercise) also shows that

$$\sum_{\xi \in \mathbf{Z}_p} |\hat{f}_{U^\perp}(\xi)|^3 = O(\delta^3) \tag{10.13}$$

and hence by Hölder's inequality we have

$$\sum_{\xi \in \mathbf{Z}_p} |\hat{f}_U(\xi)|^3 = \Omega(\delta^3). \tag{10.14}$$

Now suppose for contradiction that

$$\sum_{\xi \in S} |\hat{f}_U(\xi)|^2 \leq c\delta^2 |S|^{1/5}$$

for all sets S and some small $c > 0$ to be chosen later. Applying this in particular to the set $S = \{\xi : \hat{f}_U(\xi) \geq \lambda\}$ for some arbitrary parameter λ, we see that

$$\lambda^2 |\{\xi : \hat{f}_U(\xi) \geq \lambda\}| \leq c\delta^2 |\{\xi : \hat{f}_U(\xi) \geq \lambda\}|^{1/5}$$

[1] We thank Ben Green for presenting these arguments to the authors.

and hence

$$|\{\xi : |\hat{f}_U(\xi)| \geq \lambda\}| \leq c^{5/4}\delta^{5/2}\lambda^{-5/2}.$$

Multiplying this by $3\lambda^2$ and integrating we obtain

$$3\int_0^\delta |\{\xi : |\hat{f}_U(\xi)| \geq \lambda\}|\lambda^2\, d\lambda = O\big(c^{5/4}\delta^3\big).$$

But one can easily verify (e.g. using (4.15)) that $|\hat{f}_U(\xi)| \leq \delta$, and so the left-hand side simplifies to $\sum_{\xi\in\mathbf{Z}_p} |\hat{f}_U(\xi)|^3$. But this will contradict (10.14) if c is sufficiently small. The claim follows. □

Now we need the following variant of Lemma 10.25.

Lemma 10.29 (Non-uniformity implies density increment) *[287] Let N and p be as in the preceding lemma. Let $f : \mathbf{Z}_p \to \mathbf{R}$ be a function supported on $[1, n]$ such that $|f(n)| \leq 1$ for all n, and such that*

$$\sum_{\xi\in S} |\hat{f}(\xi)|^2 \geq \sigma \tag{10.15}$$

for some set $S \subset \mathbf{Z}_p$ and some $\sigma > 0$. Then there exists a proper arithmetic progression $P' \subset [1, N]$ with $|P'| = \Omega(\sigma N^{1/(|S|+1)})$ and

$$|\mathbf{E}_{n\in P'} f(n)| = \Omega(\sigma).$$

Proof By Kronecker's theorem, we can find $1 \leq r \leq N^{1-\frac{1}{|S|+1}}$ such that $\|r\xi\|_{\mathbf{R}/\mathbf{Z}} \leq N^{-\frac{1}{|S|+1}}$ for all $\xi \in S$. Let Q be the progression $Q = [1, N^{\frac{1}{|S|+1}}/10] \cdot r$, then a simple computation shows that

$$\frac{1}{\mathbf{P}_{\mathbf{Z}_p}(Q)}|\hat{1}_Q(\xi)| = \Theta(1) \text{ for all } \xi \in S.$$

In particular, from (4.2), (4.9), (10.15) and the previous line we have

$$\left\| f * \frac{1}{\mathbf{P}_{\mathbf{Z}_p}(Q)} 1_Q \right\|_{L^2(Z)}^2 = \sum_{\xi\in\mathbf{Z}_p} \frac{1}{\mathbf{P}_{\mathbf{Z}_p}(Q)^2} |\hat{1}_Q(\xi)|^2 |\hat{f}(\xi)|^2 = \Omega(\sigma).$$

On the other hand, from the boundedness of f we have

$$\left\| f * \frac{1}{\mathbf{P}_{\mathbf{Z}_p}(Q)} 1_Q \right\|_{L^1(Z)} \leq \|f\|_{L^1(Z)} \left\| \frac{1}{\mathbf{P}_{\mathbf{Z}_p}(Q)} 1_Q \right\|_{L^1(Z)} \leq 1.$$

Hence by Hölder's inequality we have

$$\left\| f * \frac{1}{\mathbf{P}_{\mathbf{Z}_p}(Q)} 1_Q \right\|_{L^\infty(Z)} = \Omega(\sigma).$$

Thus there exists $x \in Z$ such that

$$|\mathbf{E}_{y \in x-Q} f(y)| = \Omega(\sigma).$$

Setting $P' = [1, N] \cap (x - Q)$, the claim follows. □

The rest of the proof is similar to the arguments in the previous section and is left as an exercise. □

A further refinement was achieved by Bourgain [39], dispensing with the need for Kronecker's theorem altogether. The idea was to avoid using arithmetic progressions, but work entirely with Bohr sets, and in particular with *regular* Bohr sets. As a consequence, the following result was obtained, which seems to be very close to the limit of the Fourier-analytic method (it is in some sense the natural generalization of Proposition 10.12):

Theorem 10.30 (Roth–Bourgain theorem) *For additive groups Z of large finite odd order, we have $r_3(Z) = O(\sqrt{\frac{\log\log|Z|}{\log|Z|}}|Z|)$. In particular for all large N $r_3(\mathbf{Z}_N), r_3([1, N]) = O(\sqrt{\frac{\log\log N}{\log N}}N)$.*

This theorem follows easily from the following variant, which can be viewed as a generalization of Theorem 10.17:

Theorem 10.31 *For all additive groups Z of large finite odd order, and all $f : Z \to \mathbf{R}^+$ with $0 \leq f(x) \leq 1$, we have $\Lambda_3(f, f, f) = \Omega(\mathbf{E}_Z(f)^{-O(\mathbf{E}_Z(f)^2)})$.*

We leave the deduction of Theorem 10.30 from Theorem 10.31 as an exercise to the reader. To prove Theorem 10.31, the main tool shall be the following result, which is a substitute for Corollary 10.26.

Proposition 10.32 (Lack of progressions implies density increment) *Let Z be an additive group of large odd order, let $\mathrm{Bohr}(S, \rho)$ be a regular Bohr set of rank d, and let $f : Z \to \mathbf{R}^+$ be such that $0 \leq f(x) \leq 1$ and $\mathbf{E}_{x \in x_0 + \mathrm{Bohr}(S,\rho)} f(x) \geq \delta$ for some $x_0 \in Z$. Suppose also that*

$$\Lambda_3(f, f, f) \leq \left(\left(\frac{\delta}{2d}\right)^{100} \rho\right)^d.$$

Then there exists a regular Bohr set $\mathrm{Bohr}(S', \rho')$ of rank at most $d + 1$ and radius $\rho' \geq (\frac{\delta}{2d})^{31}\rho$ and an element $x_0' \in Z$ such that

$$\mathbf{E}_{x \in x_0' + \mathrm{Bohr}(S',\rho')} f(x) \geq \delta + \delta^2/2^{10}.$$

The deduction of Theorem 10.31 from Proposition 10.32 is also straightforward, and is left as another exercise to the reader.

Proof By translation we may take $x_0 = 0$. By increasing δ if necessary we may assume $\mathbf{E}_{x \in \mathrm{Bohr}(S,\rho)} f(x) = \delta$. By reducing f to zero outside of $\mathrm{Bohr}(S, \rho)$, we may assume that f is supported on $\mathrm{Bohr}(S, \rho)$. Now suppose for sake of contradiction that

$$\mathbf{E}_{x \in x_0' + \mathrm{Bohr}(S', \rho')} f(x) < \delta + \delta^2/2^{10} \tag{10.16}$$

for all $x_0' \in Z$ and all Bohr sets $\mathrm{Bohr}(S', \rho')$ of rank at most $d+1$ and radius at least $(\frac{\delta}{2d})^{31}\rho$.

By Lemma 4.25 we can find $0 < \rho_3 < \rho_2 < \rho_1 < \rho$ such that for each $j = 1, 2, 3$, we have[1]

$$\left(\frac{\delta}{2d}\right)^{10j+1} \rho \leq \rho_j \leq \left(\frac{\delta}{2d}\right)^{10j} \rho$$

and that $\mathrm{Bohr}(S, \rho_j)$ is regular. Note that the sets $\mathrm{Bohr}(S, \rho)$, $\mathrm{Bohr}(S, \rho_1)$, $\mathrm{Bohr}(S, \rho_2)$, $\mathrm{Bohr}(S, \rho_3)$ will differ in size by factors of $\delta^{O(d)}$, which will be too large for our application. Hence we shall have to keep careful track of the densities of each of these Bohr sets separately.

By hypothesis and a change of variable, we have

$$\mathbf{E}_{x,r \in Z} f(x-r) f(x) f(x+r) = \Lambda_3(f, f, f) \leq \left(\left(\frac{\delta}{2d}\right)^{100} \rho\right)^d;$$

in particular, from (4.25) we have

$$\mathbf{E}_{x,r \in Z} f(x-r) f(x) f(x+r) \leq \frac{\delta^3}{4} \mathbf{P}_Z(\mathrm{Bohr}(S, \rho)) \mathbf{P}_Z(\mathrm{Bohr}(S, \rho_1))$$

(say). Since f is non-negative, we can localize r to $\mathrm{Bohr}(S, \rho_1)$ and conclude

$$\mathbf{E}_{x \in Z; r \in \mathrm{Bohr}(S, \rho_1)} f(x-r) f(x) f(x+r) \leq \frac{\delta^3}{4} \mathbf{P}_Z(\mathrm{Bohr}(S, \rho)). \tag{10.17}$$

Write $f_{U^\perp} := \delta 1_{\mathrm{Bohr}(S,\rho)}$. From the symmetry of the above expression in r, one can verify the identity

$$\begin{aligned} &\mathbf{E}_{x \in Z; r \in \mathrm{Bohr}(S, \rho_1)} f(x-r) f(x) f(x+r) \\ &\quad = \mathbf{E}_{x \in Z; r \in \mathrm{Bohr}(S, \rho_1)} f_{U^\perp}(x-r) f(x) f_{U^\perp}(x+r) \\ &\qquad + \mathbf{E}_{x \in Z; r \in \mathrm{Bohr}(S, \rho_1)} (f - f_{U^\perp})(x-r) f(x) (f + f_{U^\perp})(x+r). \end{aligned} \tag{10.18}$$

[1] The reader should not take the numerical quantities (especially the powers of 2) too seriously in this argument; they are certainly not optimal.

Observe that if $x \in \mathrm{Bohr}(S, \rho - \rho_1)$ and $r \in \mathrm{Bohr}(S, \rho_1)$, then $x \pm r \in \mathrm{Bohr}(S, \rho)$, and therefore

$$\mathbf{E}_{r \in \mathrm{Bohr}(S,\rho_1)} f_{U^\perp}(x - r) f(x) f_{U^\perp}(x + r) = \delta^2 f(x).$$

Thus by positivity of f and $f_{U^\perp}$

$$\mathbf{E}_{x \in Z; r \in \mathrm{Bohr}(S,\rho_1)} f_{U^\perp}(x - r) f(x) f_{U^\perp}(x + r) \geq \delta^2 \mathbf{E}_{x \in Z} f(x) 1_{\mathrm{Bohr}(S,\rho-\rho_1)}(r).$$

By hypothesis we have

$$\mathbf{E}_{x \in Z} f(x) 1_{\mathrm{Bohr}(S,\rho)}(r) = \delta \mathbf{P}_Z(\mathrm{Bohr}(S, \rho))$$

while from regularity of $\mathrm{Bohr}(S, \rho)$ we have

$$\mathbf{E}_{x \in Z} f(x) 1_{\mathrm{Bohr}(S,\rho-\rho_1)}(r) \leq \frac{\delta}{2} \mathbf{P}_Z(\mathrm{Bohr}(S, \rho))$$

(say). Combining the above three estimates we obtain

$$\mathbf{E}_{x \in Z; r \in \mathrm{Bohr}(S,\rho_1)} f_{U^\perp}(x - r) f(x) f_{U^\perp}(x + r) 1_{\mathrm{Bohr}(S,\rho_1)}(r) \geq \frac{\delta^3}{2} \mathbf{P}_Z(\mathrm{Bohr}(S, \rho))$$

Combining this with (10.17), (10.18) we conclude that

$$|\mathbf{E}_{x \in Z; r \in \mathrm{Bohr}(S,\rho_1)} (f - f_{U^\perp})(x - r) f(x) (f + f_{U^\perp})(x + r)| \geq \frac{\delta^3}{4} \mathbf{P}_Z(\mathrm{Bohr}(S, \rho));$$

we shift this by r to obtain

$$|\mathbf{E}_{x \in Z; r \in \mathrm{Bohr}(S,\rho_1)} (f - f_{U^\perp})(x) f(x + r) (f + f_{U^\perp})(x + 2r)| \geq \frac{\delta^3}{4} \mathbf{P}_Z(\mathrm{Bohr}(S, \rho));$$

We would like to use this fact to deduce some linear bias in $f - f_{U^\perp}$. Unfortunately the constraint $r \in \mathrm{Bohr}(S, \rho_1)$ is not favorable (it localizes r to a smaller scale than x). To resolve this we need to localize the x variable to a smaller scale, namely ρ_2. To do this we write $x = y + z$ where z is restricted to $\mathrm{Bohr}(S, \rho_2)$, and conclude that

$$|\mathbf{E}_{y \in Z; r \in \mathrm{Bohr}(S,\rho_1); z \in \mathrm{Bohr}(S,\rho_2)} (f - f_{U^\perp})(y + z) f(y + z + r) (f + f_{U^\perp})(y + z + 2r)|$$
$$\geq \frac{\delta^3}{4} \mathbf{P}_Z(\mathrm{Bohr}(S, \rho)).$$

Observe that we may localize y to $\mathrm{Bohr}(S, \rho + \rho_2)$ since the expression inside the expectation vanishes otherwise. Since f is bounded and $\mathrm{Bohr}(S, \rho)$ is regular, the contribution of $\mathrm{Bohr}(S, \rho + \rho_2) \backslash \mathrm{Bohr}(S, \rho)$ can be crudely bounded by

$$\mathbf{P}_Z(\mathrm{Bohr}(S, \rho + \rho_2)) - \mathbf{P}_Z(\mathrm{Bohr}(S, \rho)) \leq \frac{\delta^3}{8} \mathbf{P}_Z(\mathrm{Bohr}(S, \rho - \rho_2))$$

(say). Thus we can restrict y to $\mathrm{Bohr}(S, \rho - \rho_2)$ and use the triangle inequality to obtain

$$\mathbf{E}_{y \in \mathrm{Bohr}(S,\rho-\rho_2)} F(y) \geq \frac{\delta^3}{8}$$

where

$$F(y) := |\mathbf{E}_{r \in \mathrm{Bohr}(S,\rho_1); z \in \mathrm{Bohr}(S,\rho_2)} (f - f_{U^\perp})(y+z) f(y+z+r)(f + f_{U^\perp})(y+z+2r)|.$$

Now that the position variable z is localized to a smaller scale than the shift variable r we may now remove the shift restriction $r \in \mathrm{Bohr}(S, \rho_1)$ as follows. We rewrite

$$F(y) = \frac{\left|\mathbf{E}_{z \in \mathrm{Bohr}(S,\rho_2)} \mathbf{E}_{r \in Z} 1_{\mathrm{Bohr}(S,\rho_1)}(r)(f - f_{U^\perp})(y+z) f(y+z+r)(f + f_{U^\perp})(y+z+2r)\right|}{\mathbf{P}_Z(\mathrm{Bohr}(S, \rho_1))}.$$

Now note that for each fixed y and each fixed $z \in \mathrm{Bohr}(S, \rho_2)$, the function

$$1_{\mathrm{Bohr}(S,\rho_1)}(r) - 1_{y+\mathrm{Bohr}(S,\rho_1)}(y+z+r) 1_{2 \cdot \mathrm{Bohr}(S,\rho_1)}(y+z+2r)$$

has an $L^1(Z)$ norm in the r variable of at most $\mathbf{P}_Z(\mathrm{Bohr}(S, \rho_1 + 2\rho_2) \backslash \mathrm{Bohr}(S, \rho_1 - 2\rho_2))$, which by the regularity of $\mathrm{Bohr}(S, \rho_1)$ will be at most $\frac{\delta^3}{16} \mathbf{P}_Z(\mathrm{Bohr}(S, \rho_1))$. Using this and the boundedness of f, we see that if we write

$$\begin{aligned} \tilde{F}(y) := &\frac{1}{\mathbf{P}_Z(\mathrm{Bohr}(S, \rho_1)} |\mathbf{E}_{z \in \mathrm{Bohr}(S,\rho_2)} \mathbf{E}_{r \in Z} \\ & 1_{y+\mathrm{Bohr}(S,\rho_1)}(y+z+r) 1_{2 \cdot \mathrm{Bohr}(S,\rho_1)}(y+z+2r) \\ & (f - f_{U^\perp})(y+z) f(y+z+r)(f + f_{U^\perp})(y+z+2r)| \\ = &\frac{|\Lambda_3((f - f_{U^\perp}) 1_{y+\mathrm{Bohr}(S,\rho_2)}, f 1_{y+\mathrm{Bohr}(S,\rho_1)}, (f + f_{U^\perp}) 1_{y+2 \cdot \mathrm{Bohr}(S,\rho_1)})|}{\mathbf{P}_Z(\mathrm{Bohr}(S, \rho_1) \mathbf{P}_Z(\mathrm{Bohr}(S, \rho_2))}, \end{aligned}$$

then $F(y)$ and $\tilde{F}(y)$ differ by at most $\delta^3/16$. In particular we have

$$\mathbf{E}_{y \in \mathrm{Bohr}(S,\rho-\rho_2)} \tilde{F}(y) \geq \frac{\delta^3}{16} \quad (10.19)$$

At this point we need to pause to address a technical issue, namely that the function $(f - f_{U^\perp}) 1_{y+\mathrm{Bohr}(S,\rho_2)}$ may have non-zero mean. Fortunately this can be dealt with by the first moment method. Let $G(y)$ denote the function

$$G(y) := \mathbf{E}_{x \in y+\mathrm{Bohr}(S,\rho_2)} (f - f_{U^\perp}).$$

Since f and $f_{U^\perp}$ range between 0 and 1 and have the same mean, we see that G is bounded in magnitude by 1 and has mean zero. Also, $G(y)$ vanishes when $y \in \mathrm{Bohr}(S, \rho + \rho_2)$, while from (10.16) we see that $G(y)$ is bounded above by

$\delta^2/2^{10}$ when $y \in \mathrm{Bohr}(S, \rho - \rho_2)$. Since $\mathrm{Bohr}(S, \rho)$ is regular, we thus see that

$$\begin{aligned}\mathbf{E}_{x\in Z} \max(G(y), 0) &\le \frac{\delta^2}{2^{10}}\mathbf{P}_Z(\mathrm{Bohr}(S, \rho - \rho_2))\\ &\quad + \mathbf{P}_Z(\mathrm{Bohr}(S, \rho + \rho_2)\backslash\mathrm{Bohr}(S, \rho - \rho_2))\\ &\le \frac{\delta^2}{2^9}\mathbf{P}_Z(\mathrm{Bohr}(S, \rho - \rho_2)).\end{aligned}$$

Since $|G(y)| = G(y) + 2\max(G(y), 0)$, we thus have

$$\mathbf{E}_{x\in\mathrm{Bohr}(S,\rho-\rho_2)}|G(y)| \le \frac{1}{\mathbf{P}_Z(\mathrm{Bohr}(S, \rho - \rho_1))}\mathbf{E}_{x\in Z}|G(y)| \le \frac{\delta^2}{2^8};$$

we can combine this with (10.19) to obtain

$$\mathbf{E}_{y\in\mathrm{Bohr}(S,\rho-\rho_2)}\tilde{F}(y) - 8\delta|G(y)| \ge \frac{\delta^3}{32}$$

and thus there exists $y \in \mathrm{Bohr}(S, \rho - \rho_2)$ such that

$$\tilde{F}(y) \ge 8\delta|G(y)| + \frac{\delta^3}{32}.$$

We fix this y and return to the analysis of $\tilde{F}(y)$. From Proposition 10.11 we have

$$\begin{aligned}\tilde{F}(y) \le &\frac{1}{\mathbf{P}_Z(\mathrm{Bohr}(S, \rho_1)\mathbf{P}_Z(\mathrm{Bohr}(S, \rho_2))}\|(f - f_{U^\perp})1_{y+\mathrm{Bohr}(S,\rho_2)}\|_{u^2(Z)}\\ &\times \|f1_{y+\mathrm{Bohr}(S,\rho_1)}\|_{L^2(Z)}\|(f + f_{U^\perp})1_{y+2\cdot\mathrm{Bohr}(S,\rho_1)}\|_{L^2(Z)}.\end{aligned}$$

From (10.16) we have

$$\|f1_{y+\mathrm{Bohr}(S,\rho_1)}\|^2_{L^2(Z)} \le 2\delta\mathbf{P}_Z(\mathrm{Bohr}(S, \rho_1)))$$

and

$$\|(f + f_{U^\perp})1_{y+2\cdot\mathrm{Bohr}(S,\rho_1)}\|^2_{L^2(Z)} \le 8\delta\mathbf{P}_Z(\mathrm{Bohr}(S, \rho_1))).$$

Thus we have

$$\tilde{F}(y) \le \frac{4\delta}{\mathbf{P}_Z(\mathrm{Bohr}(S, \rho_2))}\sup_{\xi\in Z}|[(f - f_{U^\perp})1_{y+\mathrm{Bohr}(S,\rho_2)}]^\wedge(\xi)|.$$

Thus there exists $\xi \in Z$ such that

$$\frac{1}{\mathbf{P}_Z(\mathrm{Bohr}(S, \rho_2)}|[(f - f_{U^\perp})1_{y+\mathrm{Bohr}(S,\rho_2)}]^\wedge(\xi)| \ge 2|G(y)| + \frac{\delta^2}{128}.$$

Since $y \in \mathrm{Bohr}(S, \rho - \rho_2)$, we have $f_{U^\perp} = \delta$ on $y + \mathrm{Bohr}(S, \rho_2)$. We can therefore find a phase $\theta \in \mathbf{R}/\mathbf{Z}$ such that

$$\mathrm{Re}\mathbf{E}_{x\in y+\mathrm{Bohr}(S,\rho_2)}(f(x) - \delta)e(-\xi\cdot x + \theta) \ge 2|\mathbf{E}_{x\in y+\mathrm{Bohr}(S,\rho_2)}(f - \delta)| + \frac{\delta^2}{128}.$$

In particular, by the triangle inequality we have

$$\mathbf{E}_{x\in y+\mathrm{Bohr}(S,\rho_2)}(f(x)-\delta)[2+\mathrm{Re}\, e(-\xi\cdot x+\theta)] \geq \frac{\delta^2}{128}.$$

The only remaining task is to eradicate the multiplier $2+\mathrm{Re}\ e(-\xi\cdot x+\theta)$. This shall be done by replacing the Bohr set $\mathrm{Bohr}(S,\rho_2)$ with the narrower one $\mathrm{Bohr}(S',\rho_3)$, where $S' := S\cup\{\xi\}$. Writing $x = w+z$ where $z\in\mathrm{Bohr}(S',\rho_3)$, we see that

$$\mathbf{E}_{w\in Z;z\in\mathrm{Bohr}(S',\rho_3)}1_{\mathrm{Bohr}(S,\rho_2)}(w+z)(f(w+z)-\delta)$$
$$[2+\mathrm{Re}\, e(-\xi\cdot w+\theta)e(-\xi\cdot z)] \geq \frac{\delta^2}{128}\mathbf{P}_Z(\mathrm{Bohr}(S,\rho_2)).$$

Since $z\in\mathrm{Bohr}(S',\rho_3)$, we have $|e(-\xi\cdot z)-1|\leq 2\pi\rho_3$ by (4.24). It is then easy to replace $e(-\xi\cdot z)$ by 1 incurring an error of at most $\frac{\delta^2}{512}\mathbf{P}_Z(\mathrm{Bohr}(S,\rho_2))$ (say), concluding that

$$\mathbf{E}_{w\in Z;z\in\mathrm{Bohr}(S',\rho_3)}1_{\mathrm{Bohr}(S,\rho_2)}(w+z)(f(w+z)-\delta)$$
$$[2+\mathrm{Re}\, e(-\xi\cdot w+\theta)] \geq \frac{3\delta^2}{512}\mathbf{P}_Z(\mathrm{Bohr}(S,\rho_2)).$$

A similar argument (exploiting the regularity of $\mathrm{Bohr}(S,\rho_2)$) allows one to replace the cut-off $1_{\mathrm{Bohr}(S,\rho_2)}(w+z)$ by $1_{\mathrm{Bohr}(S,\rho_2)}(w)$, to obtain

$$\mathbf{E}_{w\in Z;z\in\mathrm{Bohr}(S',\rho_3)}1_{\mathrm{Bohr}(S,\rho_2)}(w)(f(w+z)-\delta)$$
$$[2+\mathrm{Re}\, e(-\xi\cdot w+\theta)] \geq \frac{\delta^2}{256}\mathbf{P}_Z(\mathrm{Bohr}(S,\rho_2))$$

which we rewrite as

$$\mathbf{E}_{w\in\mathrm{Bohr}(S,\rho_2)}[2+\mathrm{Re}\, e(-\xi\cdot w+\theta)](\mathbf{E}_{x\in w+\mathrm{Bohr}(S',\rho_3)}f(x)-\delta) \geq \frac{\delta^2}{256}.$$

On the other hand, from (10.16) and the bound $2+\mathrm{Re}\ e(-\xi\cdot w+\theta)\leq 3$ the left-hand side is bounded by $3\frac{\delta^2}{2^{10}}$, a contradiction. □

Exercises

10.4.1 Prove (10.13).

10.4.2 Complete the proof of Theorem 10.27 given Proposition 10.28 and Lemma 10.15.

10.4.3 Deduce Theorem 10.30 from Theorem 10.31.

10.4.4 Deduce Theorem 10.31 from Proposition 10.32. (Hint: use an iteration argument with about $O(1/\mathbf{E}_Z(f))$ steps, with parameter sizes $\delta = \Omega(\mathbf{E}_Z(f))$, $d = O(1/\mathbf{E}_Z(f))$ and $\rho = \Omega(\mathbf{E}_Z(f)^{O(1/\mathbf{E}_Z(f))})$ throughout the iteration.)