

- 11.2.5 Show that the  $u^d(Z)$  norm enjoys the same invariances that the  $U^d(Z)$  norm did in Exercises 11.1.8, 11.1.9, 11.1.10, as well as an analog of (11.7). Show that the more general  $u^d(B)$  norms also obey a suitable analog of Exercises 11.1.9, 11.1.10.
- 11.2.6 [160] Let  $F = F_p$  be a finite field of odd prime order, and let  $Z$  be a finite-dimensional vector space over  $F$ , with the usual bilinear form. Show that if  $\phi : Z \rightarrow \mathbf{R}/\mathbf{Z}$  is a quadratic phase function, then we have the representation  $\phi(x) = Mx \cdot x + \xi \cdot x + c$  for some unique  $c \in \mathbf{R}/\mathbf{Z}$ ,  $\xi \in Z$ , and a self-adjoint  $F$ -linear operator  $M : Z \rightarrow Z$ . Conversely, every function of this form is a quadratic phase function. What happens if  $F = F_2$  has order 2?
- 11.2.7 [160] (Quadratic Hahn–Banach theorem) Let  $F$  and  $Z$  be as in the preceding exercise, and let  $Z'$  be a subspace of  $Z$ . Show that any quadratic phase function on  $Z'$  can be extended (possibly non-uniquely) to a quadratic phase function on  $Z$ . Conclude in particular that for any  $f : Z \rightarrow \mathbf{C}$  we have  $\|f\|_{U^3(Z)} \geq \|f\|_{u^3(Z)} \geq \mathbf{P}_Z(Z') \sup_{y \in Z} \|f\|_{u^3(y+Z)}$ ; this can be viewed as a kind of converse to Theorem 11.6.
- 11.2.8 Use Proposition 11.7 and (11.8) to establish (11.15).
- 11.2.9 (Van der Corput lemma) If  $1 \leq H < M$  and  $f : [0, M] \rightarrow \mathbf{C}$  is a function bounded in magnitude by 1, show that

$$\begin{aligned} |\mathbf{E}_{x \in [0, M]} f(x)| &\leq O(\mathbf{E}_{1 \leq h \leq H} |\mathbf{E}_{x \in [0, M-H]} f(x+h) \overline{f(x)}|)^{1/2} \\ &\quad + O\left(\frac{H^{1/2}}{M^{1/2}}\right) + O\left(\frac{1}{H^{1/2}}\right). \end{aligned}$$

(Hint: extend  $f$  by zero to the integers  $\mathbf{Z}$ , and obtain a preliminary upper bound of  $(\mathbf{E}_{1 \leq h \leq H} |\mathbf{E}_{x \in [0, M-H]} f(x+h)|^2)^{1/2} + O(\frac{H^{1/2}}{M^{1/2}})$ .) Compare this with Lemma 11.3.

### 11.3 Proof of Theorem 11.6

In this section we give a proof of Theorem 11.6. Let us fix  $F, Z, f, \eta$  with the above properties. The proof proceeds in several stages.

#### 11.3.1 Locating a somewhat linear phase derivative

The first step is to apply the inverse theorem (11.9) for the  $U^2(Z)$  norm. From the recursive definition of the  $U^3(Z)$  norm we have

$$\mathbf{E}_{h \in Z} \|T^h f \overline{f}\|_{U^2(Z)}^4 \geq \eta^8$$

and hence by (11.9)

$$\mathbf{E}_{h \in Z} \|T^h f \overline{f}\|_{u^2(Z)}^2 \geq \eta^8.$$

If we let  $H \subset Z$  be the set

$$\{h \in H : \|T^h f \overline{f}\|_{u^2(Z)}^2 \geq \eta^8/2\}$$

then we have

$$\mathbf{E}_{h \in Z} \|T^h f \overline{f}\|_{u^2(Z)}^2 \leq \eta^8/2 + \mathbf{P}_Z(H)$$

and hence

$$\mathbf{P}_Z(H) \geq \eta^8/2. \tag{11.17}$$

By definition of  $H$ , we can thus find a function  $\xi : H \rightarrow Z$  such that

$$|\mathbf{E}_{x \in Z} T^h f(x) \overline{f(x)} e(-\xi(h) \cdot x)|^2 \geq \eta^8/2 \tag{11.18}$$

for all  $h \in H$ . Informally, if we use  $\phi(x)$  to denote the phase of  $f(x)$ , this estimate is asserting that  $\phi(x+h) - \phi(x) - \xi(h) \cdot x$  is in some sense approximately constant in  $x$ , so that  $\phi(x+h) - \phi(x)$  is approximately linear. The challenge is thus to “integrate” this fact and conclude that  $\phi$  is somehow approximately quadratic. To do this, the first task shall be to obtain some linearity of  $\xi(h)$  (this reflects the fact that we expect the quantity  $(h \cdot \nabla)\phi$  to somehow be linear in  $h$ ). We sum the preceding expression over all  $h \in H$  using (11.17) and conclude

$$\mathbf{E}_{h \in Z} 1_H(h) |\mathbf{E}_{x \in Z} T^h f(x) \overline{f(x)} e(-\xi(h) \cdot x)|^2 \geq \eta^{16}/4.$$

Expanding this out as in Lemma 11.3 we conclude

$$|\mathbf{E}_{x,h,k \in Z} 1_H(h) T^{h+k} f(x) \overline{T^h f(x) T^k f(x)} f(x) e(\xi(h) \cdot k)| \geq \eta^{16}/4.$$

In order to focus on  $\xi$ , we suppress the explicit mention of the functions  $f$  using the  $\mathbf{b}()$  notation. After collecting some terms we obtain

$$|\mathbf{E}_{x,h,k \in Z} \mathbf{b}(x+h, k) \mathbf{b}(x, k) 1_H(h) e(\xi(h) \cdot k)| \geq \eta^{16}/4.$$

We can eliminate the  $\mathbf{b}(x, k)$  factor using Lemma 11.3, concluding that

$$|\mathbf{E}_{x,h,h_1,k \in Z} \mathbf{b}(x+h, k) \mathbf{b}(x+h+h_1, k) 1_H(h) 1_H(h+h_1) e((h_1 \cdot \nabla)\xi(h) \cdot k)| \geq \eta^{32}/16.$$

Making the substitution  $y = x+h$  and collecting some terms this becomes

$$|\mathbf{E}_{y,h,h_1,k \in Z} \mathbf{b}(y, k, h_1) 1_H(h) 1_H(h+h_1) e((h_1 \cdot \nabla)\xi(h) \cdot k)| \geq \eta^{32}/16.$$