model case of a vector space over a finite field, mainly because the inverse theorem for $U^3$ is much weaker than that for $U^2$, in particular involving an unknown space $W$ (or a Bohr set $B$), which will ultimately depend on a certain shift parameter $h$ in an unpleasant way. To prove Proposition 11.12, Gowers employed a slightly different approach, starting with the original function $f$ and taking $k-3$ "derivatives" $f \mapsto T^h f \overline{f}$ to reduce the $U^{k-1}$ norm to the $U^2$ norm. Employing the $U^2$ inverse theorem, one then obtains a $k-3$-fold derivative function $\xi(h_1, \ldots, h_{k-3})$. The strategy is then to establish some multilinearity properties of this function $\xi$ in order to execute a similar scheme to the one described above. This requires a substantial amount of new combinatorial technology, not least of which is a multilinear version of the Balog–Szemerédi–Gowers theorem, which cannot be established simply by applying the Balog–Szemerédi–Gowers theorem separately in each variable (again because of the issue that the structures obtained in this way for one variable will depend on the other variables). See [138] for details.

## Exercises

11.3.1 (Alex Samorodnitsky, private communication) Let $f : Z \to \mathbf{C}$, and let $D : Z \times Z \to \mathbf{R}^+$ denote the quantity $D(h, \xi) := |\widehat{T^h f \overline{f}}(\xi)|^2$. Establish the identity

$$\sum_{\xi_1,\xi_2,\xi_3,\xi_4 \in Z : \xi_1+\xi_2=\xi_3+\xi_4} \mathbf{E}_{h_1,h_2,h_3,h_4 \in Z : h_1+h_2=h_3+h_4} \prod_{j=1}^{4} D(h_j, \xi_j)$$
$$= \sum_{\xi \in Z} \mathbf{E}_{h \in Z} D(h, \xi)^4.$$

(Hint: first show that $D$ is essentially its own Fourier transform.) This identity can be used as a substitute for the first part of the above argument.

## 11.4 Soft obstructions to uniformity

In the last two sections we described the approach of Gowers in proving Szemerédi's theorem. There were three main components to the argument. First, there was the generalized von Neumann theorem (11.8) which showed among other things that one could approximate $\Lambda_k(f, \ldots, f)$ by $\mathbf{E}_Z(f)^k$ as long as $f - \mathbf{E}_Z(f)$ was sufficiently Gowers uniform of order $k-2$. Second, there was the inverse theorem, which implied that if $f - \mathbf{E}_Z(f)$ was not Gowers uniform of order $k-2$ then there was enough structure on $f$ to conclude a density increment for $f$ on a subspace or sub-progression of $Z$. Finally there was the standard density incrementation argument that iterated the previous two observations to conclude the proof of Szemerédi's theorem.

Of the three components mentioned above, the second was by far the most difficult. The reason is that this approach requires a rather strong type of inverse theorem, and in particular requires one to give quite "concrete" or "hard" obstructions to Gowers uniformity, in order to conclude the desired density increment. There is however an alternative approach, similar to the finitary ergodic argument given in Section 10.5, which requires much "softer" obstructions to Gowers uniformity, in the sense that these obstructions are not presented in as explicit a form as, say, a polynomial phase function. This makes the second stage of the argument immensely simpler. However, one must now make the third stage of the argument more complicated, replacing the density incrementation argument by an energy incrementation argument, and then establishing some sort of recurrence result for the soft obstructions. This last step now becomes rather difficult, for instance involving van der Waerden's theorem. One consequence of this is that the quantitative bounds obtained by this method are extremely poor. Nevertheless, this approach is quite robust, requiring very little arithmetic structure as compared with Gowers' approach.

To describe this approach to Szemerédi's theorem, let us first review the ingredients used in the finitary ergodic proof of Roth's theorem in Section 10.5. The strategy was to approximate the original function $f$ by some low complexity approximation $f_{U^\perp}$, such that the error $f_U = f - f_{U^\perp}$ was suitably uniform. One achieves this iteratively: if one has some preliminary approximation $f_{U^\perp}$ whose error $f_U$ is not sufficiently uniform, then one concludes that $f_U$ correlates with a certain obstruction to uniformity, which in this case was a character $e_\xi$. One then constructs a $\sigma$-algebra out of this obstruction $e_\xi$ and uses that algebra to refine the approximation $f_{U^\perp}$ to $f$, increasing the energy ($L^2$ norm) of $f_{U^\perp}$ in the process. One repeats this procedure until the error finally becomes uniform (and hence negligible). The only remaining task is then to establish some recurrence property for the approximation $f_{U^\perp}$, namely a lower bound on $\Lambda_k(f_{U^\perp}, \ldots, f_{U^\perp})$. The key here was that the approximation $f_{U^\perp}$ was built out of the $\sigma$-algebras associated to characters, and was hence *almost periodic*; this led to a non-trivial recurrence property for $f_{U^\perp}$.

The above argument used Fourier analysis by involving the characters $e_\xi$. However, one could replace this family of functions by any other family of functions, provided that two properties hold: firstly, that there were enough functions to provide a complete set of obstructions to Gowers uniformity of order $k - 2$, and secondly, that any function generated by these functions (or more precisely by their associated $\sigma$-algebras) had enough "almost periodicity" to lead to recurrence.

Using this observation, it becomes possible to dispense with Fourier analysis altogether by working with a somewhat different family of functions, replacing the characters with *dual functions of order* $k-1$ and almost periodic functions with *uniformly almost periodic functions of order* $k-2$.

We now discuss these concepts in more detail. We begin with the concept of a dual function.

**Definition 11.13 (Dual function)** If $f : Z \to \mathbf{C}$ and $d \geq 1$, we define the *dual function* $\mathcal{D}_d(f) : Z \to \mathbf{C}$ recursively by

$$\mathcal{D}_1(f)(x) = \mathbf{E}_Z(f); \quad \mathcal{D}_{d+1}(f)(x) = \mathbf{E}_{h \in Z} T^h f(x) \overline{\mathcal{D}_d(T^h f \overline{f})(x)}.$$

When $d = 2$ one can compute the dual function in terms of the Fourier transform:

$$\begin{aligned} \mathcal{D}_2(f)(x) &= \mathbf{E}_{h \in Z} T^h f(x) \overline{\mathbf{E}_Z(T^h f \overline{f})} \\ &= \mathbf{E}_{h,k \in Z} T^h f(x) T^k f(x) \overline{T^{h+k} f(x)} \\ &= \sum_{\xi \in Z} |\hat{f}(\xi)|^2 \hat{f}(\xi) e(\xi \cdot x). \end{aligned} \tag{11.24}$$

we leave this as an exercise. The formula for higher $d$ is more complicated, for instance

$$\mathcal{D}_3(f)(x) = \mathbf{E}_{h,k,l \in Z} T^h f(x) T^k f(x) T^l f(x) \overline{T^{h+k} f(x) T^{h+l} f(x) T^{k+l} f(x)} T^{h+k+l}(x).$$

We observe the useful translation and conjugation invariance

$$\mathcal{D}_d(T^h f) = T^h \mathcal{D}_d(f); \quad \mathcal{D}_d(\overline{f}) = \overline{\mathcal{D}_d(f)} \tag{11.25}$$

which is easily established by induction.

Dual functions are intimately connected with the Gowers uniformity norm. An easy induction gives the identity

$$\|f\|_{U^d(Z)}^{2^d} = \langle f, \mathcal{D}_d(f) \rangle_{L^2(Z)} = \mathbf{E}_{x \in Z} f(x) \overline{\mathcal{D}_d(f)(x)} \tag{11.26}$$

while from the Gowers–Cauchy–Schwarz inequality (11.6) we have the inequality

$$\left| \langle g, \mathcal{D}_d(f) \rangle_{L^2(Z)} \right| \leq \|g\|_{U^d(Z)} \|f\|_{U^d(Z)}^{2^d - 1} \tag{11.27}$$

for all $f, g : Z \to \mathbf{C}$. In particular we have the dual characterization of $U^d(Z)$:

$$\|g\|_{U^d(Z)} = \sup \left\{ \left| \langle g, \mathcal{D}_d(f) \rangle_{L^2(Z)} \right| : \|f\|_{U^d(Z)} \leq 1 \right\} \tag{11.28}$$

which explains the terminology "dual function". From (11.26) we immediately obtain an easy inverse theorem:

**Lemma 11.14 (Soft inverse theorem)** *Let* $f : Z \to \mathbf{C}$ *be a function bounded in magnitude by* 1, *and let* $F = \mathcal{D}_d(f)$ *be the dual function. If* $\|f\|_{U^d(Z)} \geq \eta$, *then* $|\langle f, F \rangle| \geq \eta^{2^d}$.

Thus dual functions are a complete set of obstructions to Gowers uniformity, and will play the role that the characters $e_\xi$ played in Section 10.5. (To see the connection, observe that $\mathcal{D}_2(e_\xi) = e_\xi$ for any character $e_\xi$, thus characters are themselves a kind of dual function.) To use this inverse theorem effectively in the finitary ergodic argument, we need to show that functions that are generated out of $\sigma$-algebras of dual functions obey some sort of "almost periodicity" property. The actual definition is rather strange-looking and to motivate it we first give an informal discussion. For sake of concreteness we work in the group $\mathbf{Z}_N$. In this setting, all functions $f : \mathbf{Z}_N \to \mathbf{C}$ are of course periodic of order $N$, but we are interested in almost periodicity properties which occur for shifts much smaller than $N$, in the sense that the shifts $T^n f$ are somehow compressed into a space of "dimension" much smaller than $N$, whatever that means. As it turns out, there will be a different notion of almost periodicity for each order $d - 1$; roughly speaking, a function should be almost periodic of order $d - 1$ if its phase or phases behave like a polynomial of degree $d - 1$.

Let us quantify this intuition with examples. The function $f(x) = e(\xi x/N)$ is a model example of a function which we expect to be "almost periodic of order 1", as its shifts $T^n f$ are quite recurrent. Indeed we have the formula

$$T^n f = c_n f$$

where $c_n$ are the constants $c_n = e(\xi n/N)$. If we instead take the function $f(x) = e(\xi_1 x/N) + e(\xi_2 x/N)$, then this function would still be considered almost periodic of order 1, since we have the formula

$$T^n f = c_{n,1} g_1 + c_{n,2} g_2$$

where $c_{n,j}$ are the constants $c_{n,j} = e(\xi_j n/N)$, and $g_j$ are the bounded functions $g_j(x) = e(\xi_j x/N)$. Thus in this case the shifts $T^n f$ of $f$ only vary in a two-dimensional space.

Next, we consider the function $f(x) = e(ax^2/N)$. This function would not be considered almost periodic in the usual sense, as the shifts seem to take values in a very high-dimensional space (as large as $N$). Indeed we have the shift formula

$$T^n f = c_n f$$

where the $c_n$ are no longer constant, but are themselves linearly independent functions of $x$: $c_n(x) = e((2anx + n^2)/N)$. However, observe that while the $c_n$ are not constant, they are still "simpler" than the original function $f$ because they are almost periodic of order 1, whereas we expect the quadratic object $f$ to be almost periodic of order 2.

One can of course continue these examples. They lead to the following recursive heuristic: a function $f$ should be considered almost periodic of order $d - 1$ if one has some representation of the form $T^n f = c_{n,1} g_1 + c_{n,2} g_2 + \cdots$, where the $g_1, g_2, \ldots$ are bounded functions and the $c_{n,1}, c_{n,2}, \ldots$ are almost periodic of order $d - 2$. Of course one should also provide some bound as to how many terms appear in this expansion, otherwise everything will be almost periodic of every order.

A convenient way to formalize the above intuition is as follows.

**Definition 11.15 (Uniform almost periodicity norms)** [357] If $f : Z \to \mathbf{C}$, we define $\|f\|_{UAP^0(Z)}$ to be infinite if $f$ is non-constant, and equal to $|c|$ if $f$ is equal to a constant $c$. If we now inductively assume that the $UAP^d(Z)$ norm has been defined for some $d$, we define the $UAP^{d+1}(Z)$ *norm* of $f$ to be the infimum of all the constants $M > 0$ for which one has a representation formula of the form

$$T^n F = M\mathbf{E}(c_{n,h} g_h) \text{ for all } n \in Z, \tag{11.29}$$

where $H$ is a finite non-empty set, $g = (g_h)_{h\in H}$ is a collection of functions from $Z$ to $\mathbf{C}$ with $\|g_h\|_{L^\infty(Z)} \leq 1$, $c = (c_{n,h})_{n\in Z, h\in H}$ is a collection of functions from $Z$ to $\mathbf{C}$ with $\|c_{n,h}\|_{UAP^d(Z)} \leq 1$, and $h$ is a random variable taking values in $H$.

We informally refer to a function as *uniformly almost periodic of order* $d - 1$ if its $UAP^{d-1}(Z)$ norm is bounded.

One can easily check inductively that the $UAP^d(Z)$ norms are finite for $d \geq 1$, and are indeed norms, in particular obeying the triangle inequality

$$\|f + g\|_{UAP^d(Z)} \leq \|f\|_{UAP^d(Z)} + \|g\|_{UAP^d(Z)}. \tag{11.30}$$

Moreover, we have the important *Banach algebra property*

$$\|fg\|_{UAP^d(Z)} \leq \|f\|_{UAP^d(Z)} \|g\|_{UAP^d(Z)}. \tag{11.31}$$

We leave the easy verification of these facts as an exercise; the rather complicated construction in Definition 11.15 was designed primarily in order to obtain these nice properties (11.30), (11.31).

The $UAP^{d-1}$ norms are a kind of dual to the $U^d$ norms; see Exercise 11.4.8. The $UAP^1$ norm is the same as the Wiener algebra norm, see Exercise 11.4.10. They are also connected to dual functions:

**Lemma 11.16** *Let* $f : Z \to \mathbf{C}$ *be a function bounded in magnitude by* 1. *Then* $\|\mathcal{D}_d(f)\|_{UAP^{d-1}(Z)} \leq 1$ *for all* $d \geq 1$.

*Proof* We induce on $d$. The case $d = 1$ is clear. Now suppose that $d \geq 2$ and the claim has already been proven for $d - 1$. From the definition of $\mathcal{D}_d(f)$ and (11.25), and the change of variables $n + h = h'$, we have

$$T^n \mathcal{D}_d(f) = \mathbf{E}_{h\in Z}(T^{n+h} f \overline{\mathcal{D}_{d-1}(T^{n+h} f \overline{T^n f})}) = \mathbf{E}_{h'\in Z}(\mathcal{D}_{d-1}(\overline{T^{h'} f} T^n f) T^{h'} f).$$

The claim then follows by setting $M := 1$, $H := Z$, $c_{n,h} = \mathcal{D}_{d-1}(\overline{T^h f T^n f})$, and $g_h := T^h f$. □

Combining this with Lemma 11.14 we see that the uniformly almost periodic functions of order $d-1$ form a complete set of obstructions for the Gowers uniformity norm of order $d$:

**Corollary 11.17 (Soft inverse theorem, II)** *Let* $f : Z \to \mathbf{C}$ *be a function bounded in magnitude by* 1 *with* $\|f\|_{U^d(Z)} \geq \eta$. *Then there exists a function* $F : Z \to \mathbf{C}$ *such that* $\|F\|_{UAP^{d-1}} \leq 1$ *and* $|\langle f, F\rangle| \geq \eta^{2^d}$.

One now has enough machinery to prove the following variant of Proposition 10.36.

**Proposition 11.18 (Koopman–von Neumann decomposition)** *[357] Let* $k \geq 3$, *let* $f : Z \to \mathbf{R}^+$ *be such that* $0 \leq f(x) \leq 1$, *let* $\sigma > 0$, *and let* $F : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ *be an arbitrary function. Then there exists a quantity* $K = O_{\sigma,F,k}(1)$ *and a decomposition* $f = f_{U^\perp} + f_U$ *with the following properties:*

- *the "anti-uniform" component* $f_{U^\perp}$ *obeys the bounds* $0 \leq f_{U^\perp} \leq 1$ *and* $\mathbf{E}_Z f_{U^\perp} = \mathbf{E}_Z f$. *Furthermore there exists an approximation* $f_{UAP}$ *to* $f_{U^\perp}$ *with* $0 \leq f_{UAP} \leq 1$, $\|f_{U^\perp} - f_{UAP}\|_{L^2(Z)} \leq \sigma$, *and* $\|f_{UAP}\|_{UAP^{k-2}(Z)} \leq K$;
- *the "uniform" component* $f_U$ *obeys the Gowers uniformity estimate* $\|f_U\|_{U^{k-1}(Z)} \leq \frac{1}{F(\sigma,K)}$.

This proposition is proven by almost identical means to Proposition 10.36 and we leave it as an exercise. The soft inverse theorem in Corollary 11.17 allows us to use uniformly almost periodic functions as a substitute for characters (and for quasi-periodic functions); the Banach algebra properties of such functions are the substitute for the fact that polynomial combinations of almost periodic functions are almost periodic. Otherwise the proof is much the same.

To conclude the proof of Szemerédi's theorem $r_k(\mathbf{Z}_N) = o_{N\to\infty;k}(N)$, one needs a recurrence theorem for the almost periodic component:

**Proposition 11.19 (Uniformly almost periodic functions are recurrent)** *[357] Let* $k \geq 3$, *let* $N$ *be a large prime, let* $f_{U^\perp}, f_{UAP} : \mathbf{Z}_N \to \mathbf{R}^+$ *be such that* $0 \leq f_{U^\perp}, f_{UAP} \leq 1$, $\mathbf{E}_{\mathbf{Z}_N} f_{U^\perp} \geq \delta$, $\|f_{U^\perp} - f_{UAP}\|_{L^2(\mathbf{Z}_N)} \leq \frac{\delta^2}{1024k}$, *and* $\|f_{UAP}\|_{UAP^{k-2}(\mathbf{Z}_N)} \leq K$. *Then we have*

$$\Lambda_k(f_{U^\perp}, \ldots, f_{U^\perp}) = \Omega_{k,\delta,K}(1).$$

From Proposition 11.19, Proposition 11.18 and (11.8) one can conclude Szemerédi's theorem by the same argument as in Section 10.5. The proof of Proposition 11.19, however, is rather difficult, invoking an induction on $k$, the use of an energy increment argument to regularize certain $\sigma$-algebras which will appear, some Hilbert space arguments to locally compactify shift orbits such as $\{T^n f_{UAP} : f_{UAP} \in \mathbf{Z}_N\}$, and then van der Waerden's theorem to find monochromatic arithmetic progressions, where the coloring is determined by the local compactification. We will not prove it in full generality here, referring the reader to [357] for full details. However, we will sketch the somewhat simpler $k = 3$ version of the argument below. In this case one could instead rely on Exercise 11.4.10 and Proposition 10.35 to obtain a simpler proof with much more efficient bounds, but the argument we give below does not require the Fourier transform and can be extended (with additional arguments) to the higher $k$ case.

*Proof of Proposition 11.19 in the $k = 3$ case* (Sketch) We consider the shifts $\{T^n f_{UAP} : n \in \mathbf{Z}_N\}$ as a subset of $L^2(\mathbf{Z}_N)$. Since $\|f_{UAP}\|_{UAP^1} \leq K$, we see that there exists a random variable $h$ taking values in a finite set $H$ and functions $g_h : \mathbf{Z} \to \mathbf{C}$ with $\|g_h\|_{L^\infty(\mathbf{Z}_N)} \leq 1$, such that all the shifts $T^n f_{UAP}$ are contained in the set

$$\Gamma := \{K\mathbf{E}_h(c_h g_h) : c_h \in \mathbf{C}, |c_h| \leq 1 \text{ for all } h \in H\} \qquad (11.32)$$

which can be thought of as a kind of high-dimensional cube. It turns out that this set is "compact" in the sense that it can be covered by $O_{k,\delta,K}(1)$ balls in $L^2(\mathbf{Z}_N)$ of radius $\delta^2/1024$ (see Exercise 11.4.13). This induces a coloring of $\mathbf{Z}_N$ by $O_{\delta,K}(1)$ colors, by assigning to each $n \in \mathbf{Z}_N$ one of the balls that contains $T^n f$. By van der Waerden's theorem (Exercise 6.3.9), we conclude that for $\Omega_{\delta,K}(1)$ of the pairs $(a, r) \in \mathbf{Z}_N$, the triple $a, a + r, \ldots, a + 2r$ are monochromatic, so that the functions $T^a f_{UAP}$, $T^{a+r} f_{UAP}$, $T^{a+2r} f_{UAP}$ lie in the same $\delta^2/1024$-ball. This implies that the functions $T^a f_{U^\perp}$, $T^{a+r} f_{U^\perp}$, $T^{a+2r} f_{U^\perp}$ are distance at most $\delta^2/512$ apart. Since these functions are also bounded between 0 and 1 and have mean $\delta$, an application of Markov's inequality then shows that these functions are simultaneously greater than $\delta/4$ (say) on a set of density at least $\delta/4$. Thus $\mathbf{E}(T^a f_{U^\perp} T^{a+r} f_{U^\perp} T^{a+2r} f_{U^\perp}) = \Omega_\delta(1)$ for all such pairs $(a, r)$. Taking averages over all $a, r$ we obtain the claim. □

## Exercises

11.4.1 Prove (11.24) and (11.25).

11.4.2 Prove (11.26), (11.27), and (11.28).

11.4.3 Verify that $\|f\|_{UAP^d(Z)}$ is well-defined and finite for all $d \geq 1$, and obeys (11.30) and (11.31). In particular, verify that the $UAP^d(Z)$ norm is indeed a norm.

11.4.4 Establish the monotonicity property $\|f\|_{UAP^{d+1}(Z)} \leq \|f\|_{UAP^d(Z)}$ for all $f : Z \to \mathbf{C}$ and $d \geq 0$.

11.4.5 Let $\phi : Z \to Z'$ be a Freiman isomorphism of order 2. Show that $\|f \circ \phi\|_{UAP^{d-1}(Z)} = \|f\|_{UAP^{d-1}(Z')}$ for all $f : Z' \to \mathbf{C}$ and $d \geq 1$. In particular, the $UAP^{d-1}(Z)$ norms are translation-invariant.

11.4.6 Let $\phi : Z \to \mathbf{R}/\mathbf{Z}$ be a phase polynomial of degree less than $d$. Show that $\|e(\phi)f\|_{UAP^{d-1}(Z)} = \|f\|_{UAP^{d-1}(Z)}$ for all $f : Z \to \mathbf{C}$.

11.4.7 Let $\phi : Z \to \mathbf{R}/\mathbf{Z}$ be a phase polynomial of degree less than $d$. Show that $\mathcal{D}_d(e(\phi)) = e(\phi)$ and $\|e(\phi)\|_{UAP^{d-1}(Z)} = 1$, thus every polynomial phase function is a dual function.

11.4.8 [357] Obtain the inequality

$$\left|\langle f, g\rangle_{L^2(Z)}\right| \leq \|f\|_{U^d(Z)}\|g\|_{UAP^{d-1}(Z)}$$

for any $d \geq 1$ and $f, g : Z \to \mathbf{C}$. (Hint: use induction on $d$.) Thus functions which are uniformly almost periodic of order $d-1$ are almost orthogonal to Gowers uniform functions of order $d$. This can be viewed as a partial converse to Corollary 11.17. Note in particular that we have

$$\|f\|^2_{L^2(Z)} \leq \|f\|_{U^d(Z)}\|f\|_{UAP^{d-1}(Z)}$$

thus a function cannot be simultaneously uniformly almost periodic and Gowers uniform without also being small.

11.4.9 Let $f, g : Z \to \mathbf{C}$ be functions bounded in magnitude by 1. Establish the inequality

$$\|fg\|^{2^d}_{U^d(Z)} \leq \|f\|_{UAP^{d-1}(Z)}\|g\|_{U^d(Z)}$$

for all $d \geq 1$. (Hint: use Lemma 11.16 applied to $fg$, together with the algebra property of $UAP^{d-1}(Z)$.)

11.4.10 (Ben Green, private communication) Show that $\|f\|_{UAP^1(Z)} = \|\hat{f}\|_{l^1(Z)}$ for all $f : Z \to \mathbf{C}$. (Hint: from Exercise 11.4.7 and the triangle inequality one can obtain the inequality $\|f\|_{UAP^1(Z)} \leq \|\hat{f}\|_{l^1(Z)}$. To obtain the other inequality, first use Plancherel's theorem to establish that $\mathbf{E}_{n,x\in Z} c_{n,h} g_h(x) b(x+n)| \leq 1$ whenever $c_{n,h}$ is a constant bounded by 1, $g_h$ is a function with $\|g_h\|_{L^\infty(Z)} \leq 1$, and $b$ is a function with $\|\hat{b}\|_{l^\infty(Z)} \leq 1$.)

11.4.11 [357] Prove Theorem 11.18.

11.4.12 Use Proposition 11.19, Proposition 11.18 and (11.8) to deduce that $r_k(\mathbf{Z}_N) = o_{N\to\infty;k}(N)$ for all $k \geq 1$ and all large $N$.

11.4.13 [357] Let $\Gamma$ be the set defined in (11.32). Show that given any $\varepsilon > 0$, the set $\Gamma$ can be covered by $O_{\varepsilon,K}(1)$ balls in $L^2(Z)$ of radius $\varepsilon$. (Hint: find a maximal orthonormal set $v_1, \ldots, v_J$ such that $\mathbf{E}|\langle g_h, v_j\rangle_{L^2(Z)}|^2 \geq \varepsilon^2/4$ for all $1 \leq j \leq J$, and use Bessel's inequality and linearity of expectation to obtain an upper bound on $J$. Show that the quantities $\Gamma$ stay within $\varepsilon/2$ of the $J$-dimensional space spanned by $v_1, \ldots, v_J$.)