then there exists an positive integer $l = O_\delta(1)$ such that lA contains an arithmetic progression of length $\Omega_\delta(|P|)$.

12.2.3 Using only the preceding exercise and an iteration argument, show that if P is a progression of integers of rank d and $A \subset P$ is such that $|A| \geq \delta|P|$, then there exists a positive integer $l = O_{\delta,d}(1)$ such that lA contains a progression of rank at most d and cardinality $\Omega_{\delta,d}(|P|)$.

12.2.4 Without using Theorem 3.40, show that if P is a progression of integers of rank d, then there exists a positive integer $l = O_d(1)$ such that lP contains a *proper* progression of rank at most d and cardinality $\Omega_d(|P|)$. (You may wish to use Freiman's cube lemma, Theorem 5.20 or the variant in Proposition 5.35.)

12.2.5 [351] (Filling lemma) Using only the preceding exercises, show that if P is a progression of integers of rank d and $A \subset P$ is such that $|A| \geq \delta|P|$, then there exists a positive integer $l = O_{\delta,d}(1)$ such that lA contains a proper progression of rank at most d and cardinality $\Omega_{\delta,d}(|P|)$.

12.2.6 [350] Let $P = a + [0, N] \cdot v$ be a progression of rank d. Show that if P is not proper, then $|2P| \leq (1 - \frac{1}{2^{d+1}})|[0, 2N]|$. More generally, prove that

$$|kP| \leq \frac{O(1)^d}{k}|[0, kN]|$$

for all $k \geq 1$. Thus an improper progression becomes "increasingly improper" as one dilates it.

12.2.7 Using the preceding exercise, show that if P is a proper progression of integers of rank d such that $2P$ is not proper, show that there is a positive integer $l = O_d(1)$ such that lP contains a proper progression of rank at most $d - 1$ with cardinality $\Omega_d(|P|)$.

12.2.8 Use the above exercises to give alternate proofs of Lemma 12.6 and Theorem 12.5.

12.3 Generalizations and variants

There are various extensions of Theorem 12.4 and Theorem 12.5. An easy modification of the above arguments allows one to handle distinct summands:

Theorem 12.10 *[350] For any fixed positive integer d there is a constant $C_d >$ 0 such that the following holds. Let $A_1, \ldots, A_l$ be subsets of $[1, n]$ of size m where l and m satisfy $l^d m \geq C_d n$. Then $A_1 + \cdots + A_l$ contains a progression of rank d' and volume at least $\Omega_d(l^{d'} m)$, for some integer $1 \leq d' \leq d$. In particular, $A_1 + \cdots + A_l$ contains an arithmetic progression of length at least $\Omega_d(l m^{1/d})$.*

From Lemma 5.25 we can also replace the integers by any other torsion-free additive group without difficulty. A more difficult strengthening is to work with the restricted sum sets l^*A instead of lA. It is possible to adapt the above methods to deal with this case too:

Theorem 12.11 *[350] For any fixed positive integer d there is a constant $C_d > 0$ such that the following holds. For any positive integers n and l and any set $A \subset [1, n]$ satisfying $l \leq |A|/2$ and $l^d|A| \geq C_d n$, l^*A contains a proper progression of rank d' and volume at least $\Omega_d(l^{d'}|A|)$, for some integer $1 \leq d' \leq d$. In particular l^*A contains a proper arithmetic progression of length $\Omega_d(l|A|^{1/d})$.*

If we define $f^*(m, l, n)$ to be the obvious analog of $f(m, l, n)$ with lA replaced by l^*A, then we conclude (using Lemma 12.3 for the upper bound) that $f^*(m, l, n) = \Theta_d(lm^{1/d})$ whenever $C_d \frac{n}{l^d} \leq m \geq c_d \frac{n}{l^{d-1}}$ and $m \geq 2l$. (Note that $f^*(m, l, n)$ becomes vacuous when $m < l$, so the condition $m \geq 2l$ is fairly natural.)

Theorem 12.11 is significantly harder than Theorem 12.4 and we will not present it here. Let us, however, mention an important lemma, which can be viewed as a variant of Freiman theorem for subset sums. This lemma asserts that if l^*A does not yield a proper GAP as claimed by the theorem, then A must contain a big subset which has a very rigid structure.

Lemma 12.12 *For any positive constants ν and d there are positive constants δ, α and d_1 such that the following holds. Let A be a subset of $[n]$, l be a positive integer and $n \geq f(n) \geq 1$ be a function of n such that*

$$\max\left\{\log^{10} n, (40 f(n) \log_2 n)^{1/3d}\right\} \leq l \leq |A|/2$$

and $l^d|A|f(n) \geq n$. Then one of the following two statements must hold:

- *l^*A contains a proper GAP of rank d' and volume $\Omega(l^{d'}|A|)$ for some $1 \leq d' \leq d$;*
- *there is a subset $\tilde{A}$ of A with cardinality at least $\delta|A|$ which is contained in a GAP P of rank d_1 and volume $O(|A|f(n)^{1+\nu} \log^{\alpha} n)$.*

The function $f(n)$ can be seen as a *rigidity* parameter. The closer $l^d|A|$ is to n, the more rigid is the structure of $\tilde{A}$.

The case $d = 1$ is of special importance, being a generalization of the Theorem 12.2, and we isolate it as a corollary.

Corollary 12.13 *There exists a constant $C > 0$ such that whenever $A \subset [1, n]$ and $1 \leq l \leq |A|/2$ is such that $l|A| \geq Cn$, then l^*A contains an arithmetic progression of length $cl|A|$.*

This result has the following consequence for subset sums:

Corollary 12.14 *[350] If A is a subset of $[1, n]$ of cardinality at least $C\sqrt{n}$ for a sufficiently large absolute constant $C > 0$, then the subset sums $FS(A)$ contains an arithmetic progression of length n.*

The first part of this result, with $\sqrt{n}$ replaced by $\sqrt{n \log n}$, was originally proven by Freiman [115]; we leave the deduction of this corollary from Corollary 12.13 as an exercise.

Another variant is to work in a cyclic group $\mathbf{Z}_n$ of prime order instead of in an interval $[1, n]$. Here, one can modify the preceding arguments to obtain

Theorem 12.15 *[350] For any $d \geq 1$ there exists $C_d > 0$ such that the following holds. For any additive set A in a cyclic group $\mathbf{Z}_n$ of prime order and any $l \geq 1$ with $l^{d+1}|A| \geq C_d n$ and $|A| \geq 2$, the set lA contains a proper arithmetic progression of length* $\min(n, \Omega_d(l|A|^{1/d}))$.

There are two differences between this theorem and Theorem 12.4. First the progression has length $\min(n, \Omega_d(l|A|^{1/d}))$ instead of $\Omega_d(l|A|^{1/d})$, but this is natural since lA cannot exceed n in size. Second the condition on l has been relaxed from $l^d|A| \geq C_d n$ to $l^{d+1}|A| \geq C_d n$. This is ultimately because the trivial bound $|lA| \leq ln$ which was used in the $[1, n]$ case can now be improved to the trivial bound $|lA| \leq n$. Otherwise the argument is essentially the same, and is left as an exercise.

Exercises

12.3.1 Prove Theorem 12.10. (Hint: use the tree argument with different sets at the leaves. You will need to replace the Ruzsa–Chang theorem by a different result, such as Theorem 4.43, or use the elementary approach sketched in earlier exercises.)

12.3.2 Deduce Corollary 12.14 from Corollary 12.11.

12.3.3 Let $\mathbf{Z}_n$ be a cyclic group of prime order, and let $f(m, l, \mathbf{Z}_n)$ be defined just like $f(m, l, n)$ but now with the sets A lying in $\mathbf{Z}_n$ instead of $[1, n]$. Show that $f(m, l, \mathbf{Z}_n) = n$ if and only if $l(m - 1) \geq n - 1$. (Hint: use Exercise 12.1.1.) Show also that for every $d \geq 1$ there exist constants $c_d, C_d > 0$ such that $f(m, l, n) = \Theta_d(lm^{1/d})$ whenever $C_d \frac{n}{l^{d+1}} \leq m \geq c_d \frac{n}{l^d}$.

12.3.4 Prove Theorem 12.15. (Note that the hypothesis that n is prime will prevent any "torsion" issues from arising in the progressions until the progressions become of size comparable to n, at which point one can proceed using the Cauchy–Davenport inequality instead.)