12.4 Complete and subcomplete sequences

An infinite set $A \subset \mathbf{Z}^+$ of positive integers is *complete* if its subset sums $FS(A)$ contain every sufficiently large positive integer. This notion is similar to, but distinct from, the concept of a base as studied in Chapter 1, as we allow sums of arbitrary length but require the summands to be distinct. The notion of complete sequences was introduced by Erdős in the early sixties and has since then been studied extensively by various researchers (see [89, Section 6] or [274, Section 4.3] for surveys). The center of this study is to find necessary and sufficient conditions for a sequence to be complete.

Intuitively, the denser a set is, the more likely it is to be complete. However, density alone is not sufficient; the even integers have density $1/2$ but are not complete. More generally, any set contained in an infinite arithmetic progression containing zero will not be complete. To deal with these cases, let us say that a set A is *subcomplete* if $FS(A)$ contains an infinite arithmetic progression $a + \mathbf{N} \cdot r = \{a, a + r, a + 2r, \ldots\}$. It is easy to see that these two notions are related as follows:

Lemma 12.16 *Let $A \subset \mathbf{Z}^+$ be infinite. Then A is complete if and only if A is subcomplete and $FS(A)$ intersects every infinite arithmetic progression in $\mathbf{Z}^+$.*

We leave this lemma as an exercise. The condition that $FS(A)$ intersects every infinite arithmetic progression in $\mathbf{Z}^+$ is a local condition that only depends on the residue classes that A occupies, together with their multiplicity; see exercises. In particular, this condition is typically quite easy to verify for standard bases such as the Waring bases $\mathbf{N}^\wedge k$ or the primes P. Thus we shall focus on the subcomplete property.

A simple example of Cassels [46] shows that there exist sets A of density $|A \cap [1, n]| = \Omega(n^{1/2})$ which are not subcomplete; see exercises. Remarkably, this example is sharp up to constants:

Theorem 12.17 *[350] There exists an absolute constant $C > 0$ such that every infinite set $A \subset \mathbf{Z}^+$ with $|A \cap [1, n]| \geq Cn^{1/2}$ is subcomplete. In particular, if $FS(A)$ also intersects every infinite arithmetic progression in $\mathbf{Z}^+$, then A is complete.*

We prove this result in the next section; the main tool is Corollary 12.14. The second part of this result was conjectured by Erdős [85] in 1962, and the first part by Folkman [103] in 1966. In [85] the second part was proven under the stronger hypothesis $|A \cap [1, n]| \geq Cn^{(\sqrt{5}-1)/2}$, while in [103] the first part was proven under the hypothesis $|A \cap [1, n]| \geq n^{1/2+\varepsilon}$ for any $\varepsilon > 0$ and sufficiently large n. This was lowered to $Cn^{1/2} \log^{1/2} n$ by Hegyvári [181] and Luczak and Schoen [241], using Sárkőzy's theorem (see the exercises).

There is an analog of the above results for infinite multisets $A = \{a_1, a_2, \ldots\}$ in $\mathbf{Z}^+$, where $a_1 \leq a_2 \leq \cdots$ are allowed to have repetitions, and define the finite sum sets

$$FS(A) := \left\{ \sum_{i \in I} a_i : I \subset \mathbf{Z}^+, \text{ finite} \right\}$$

in analogy with before, and define the notion of completeness and subcompleteness as above. In this case it is possible to have a density as large as $|A \cap [1, n]| = \Omega_\varepsilon(n^{1-\varepsilon})$ for any given $\varepsilon > 0$ (where of course we count multiplicity) and still not have subcompleteness (see exercises). Again, this example is basically sharp.

Theorem 12.18 *[350] There exists an absolute constant $C > 0$ such that every infinite multiset $A \subset \mathbf{Z}^+$ with $|A \cap [1, n]| \geq Cn$ is subcomplete. In particular, if $FS(A)$ also intersects every infinite arithmetic progression in $\mathbf{Z}^+$, then A is complete.*

This result was conjectured by Folkman [103], and is proven very similarly to Theorem 12.17, we leave it as an exercise for the next section where Theorem 12.17 is proved.

To end this section, let us disuss the finite version of completeness. We say that a subset A of $\mathbf{Z}_p$ (for a large prime p) is complete if $FS(A) = \mathbf{Z}_p$. Olson [265], answering a question of Erdős and Heilbronn, proved that if $|A| > 2\sqrt{p}$, then A is complete. The bound $2\sqrt{p}$ is essentially sharp. To see this, take $A = \{-k, -(k-1), \ldots, -1, 0, 1, \ldots, (k-1), k\}$, where k is the largest integer such that $\sum_{i=1}^{k} i < p/2$. Deshouillers and Freiman [70] showed that this is actually the only example, given that $|A|$ is sufficiently large. We call a set A of integers between 0 and $p-1$ *small* if

$$\sum_{a \in A} \|a/p\| < 1$$

where $\|z\|$ (as usual) is the distance from z to the closest integer. It is easy to check that a small set is not complete.

Theorem 12.19 *Let A be a subset of $\mathbf{Z}_p$ with more than $\sqrt{2p}$ elements. If A is not complete, then there is a non-zero element x of $\mathbf{Z}_p$ such that the set $x \cdot A$ is small.*

Szemerédi and Vu [349, 352] showed that it is possible to weaken the condition $|A| > \sqrt{2p}$ considerably by dropping a small subset from A.

Theorem 12.20 *Let A be a non-complete subset of $\mathbf{Z}_p$. Then there is a subset A' of A with at most $p^{.49}$ elements and a non-zero element x of $\mathbf{Z}_p$ such that $x \cdot (A \backslash A')$ is small.*