

Exercises

- 12.4.1 Show that any infinite set $A \subset \mathbf{Z}^+$ of lower density strictly greater than $1/2$ is complete.
- 12.4.2 Prove Lemma 12.16. Generalize this to multisets.
- 12.4.3 Let $A \subset \mathbf{Z}^+$ be an infinite set, and for each positive integer N let $A_N^\infty \subset \mathbf{Z}_N$ be the set of all residue classes $a \pmod N$ whose intersection with A is infinite, and let A'_N be those elements of A which do not lie in one of the residue classes in A_N^∞ (this set is automatically finite). Show that $FS(A)$ intersects every infinite arithmetic progression in \mathbf{Z}^+ if and only if $FS(A'_N) \pmod N + \langle A_N^\infty \rangle = \mathbf{Z}_N$ for all N . Note that one has $FS(A'_N) \pmod N = FS(A'_N \pmod N)$ if we view $A'_N \pmod N$ as a multiset; thus this criterion uses only the multiplicities of A modulo N rather than the actual values of A .
- 12.4.4 Consider an infinite sequence $A = \{a_1, a_2, \dots\}$. Prove that if

$$\limsup_{i \rightarrow \infty} \left(a_i - \sum_{j=1}^{i-1} a_j \right) \rightarrow \infty, \tag{12.1}$$

then A is not subcomplete.

- 12.4.5 [46],[351] Let $m := 10^4$ and let $A := \bigcup_{i=1}^\infty [m^{2^i}/4, m^{2^i}/2]$. Show that $|A \cap [1, n]| = \Omega(n^{1/2})$ for all n , and A intersects every infinite arithmetic progression in \mathbf{Z}^+ but that A is not subcomplete nor complete.
- 12.4.6 Modifying the previous example, show that for any $\varepsilon > 0$ there exists a multiset A with $|A \cap [1, n]| = \Omega_\varepsilon(n^{1-\varepsilon})$ which intersects every infinite arithmetic progression in \mathbf{Z}^+ but is not subcomplete.
- 12.4.7 [181], [241] Let $A \subset [1, n]$ be such that $|A| \geq Cn^{1/2} \log^{1/2} n$ for some large constant $C > 0$ Show that there exists $1 \leq i \leq O(\log n)$ and a set $A_i \subset [1, 2n]$ such that $2^i |A_i| = \Omega(Cn)$ and each element in A_i can be written as the sum of two distinct elements of A between 2^i and 2^{i+1} times. Use this and Theorem 12.2 to prove the first part of Corollary 12.14 with $n^{1/2}$ replaced by $n^{1/2} \log^{1/2} n$. By using the arguments of the next section, this establishes Theorem 12.17, again with $n^{1/2}$ replaced by $n^{1/2} \log^{1/2} n$.

12.5 Proof of Theorem 12.17

To prove Theorem 12.17, it is convenient to reduce the (infinitary) condition of subcompleteness to a finitary version. Let us say that a partition $A = A' \cup A''$ of a multiset A of positive integers is *good* if the following two properties hold:

- there is a number d such that $FS(A')$ contains arbitrary long arithmetic progressions with difference d ;
- let $A'' = \{b_1 \leq b_2 \leq b_3 \leq \dots\}$, then

$$\lim_{i \rightarrow \infty} \left(\sum_{j=1}^{i-1} b_j \right) - b_i = +\infty. \tag{12.2}$$

Thus A' enjoys a finitary version of subcompleteness, whereas A'' grows slower than a lacunary sequence. These two conditions imply subcompleteness:

Lemma 12.21 [351, 241, 181] *Any sequence A of positive integers which admits a good partition is subcomplete.*

Proof We begin with some reductions. First observe that we can remove finitely many elements from A'' without affecting the condition (12.2). In particular, we can remove any residue class $a + d \cdot \mathbf{Z} \bmod d$ which contains only finitely many elements from A'' .

Let $A_d \subset \mathbf{Z}_d$ be the set of residue classes mod d which intersect A'' (and thus contain infinitely many elements from A'' , by the above reduction). The group $\langle A_d \rangle$ spanned by A'' is a subgroup of \mathbf{Z}_d and thus has the form $\langle A_d \rangle = d' \cdot \mathbf{Z}_d$ for some factor d' of d . In particular we see that every element of A'' is a multiple of d' . Observe that $FS(A'')$ must intersect every residue class in $\langle A_d \rangle$. Thus there exists a finite set $B \subset A''$ such that $FS(B)$ intersects every residue class in $d' \cdot \mathbf{Z}_d$.

Let $A''' := A'' \setminus B$, thus A''' still obeys (12.2). A simple greedy algorithm argument then shows that the subset sums $FS(A''')$ are *syndetic* (has bounded gaps); more precisely, there exists $L \geq 0$ such that given any positive integer n , we can find an element $m \in FS(A''')$ such that $0 \leq n - m \leq L$. Since $FS(A''')$ consists entirely of multiples of d' , and $FS(B)$ intersects every residue class in $d' \cdot \mathbf{Z}_d$, we conclude that the set $(FS(A''') + FS(B)) \cap (d \cdot \mathbf{Z})$ is also syndetic. Since $FS(A')$ contains arbitrarily long progressions of length d , we conclude that $FS(A') + [(FS(A''') + FS(B)) \cap (d \cdot \mathbf{Z})]$ contains an infinite progression of length d . But since this set is contained in $FS(A)$, we see that A is subcomplete as claimed. \square

We can now prove Theorem 12.17.

Proof of Theorem 12.17 We write $A = \{a_1 < a_2 < \dots\}$, and split $A = A' \cup A''$ where $A' := \{a_{2m} : m \in \mathbf{Z}^+\}$ and $A'' := \{a_{2m-1} : m \in \mathbf{Z}^+\}$. It is easy to see using the hypothesis $|A \cap [1, n]| \leq Cn^{1/2}$ that the set A'' will obey (12.2) and we leave it as an exercise. Thus we only need to show that $FS(A')$ contains arbitrarily long arithmetic progressions of a fixed step d .

For each non-negative integer j , let $A'_j := \{a_{2m} : 2^j \leq m < 2^{j+1}\}$. Thus the A'_j partition A' . Also, the hypothesis $|A \cap [1, n]| \leq Cn^{1/2}$ implies that for all