sufficiently large j we have $A'_j \subset [1, n_j]$ for some $n_j = \Theta(2^{2j}/C^2)$. From Corollary 12.14 we conclude (if C is large enough) that $FS(A'_j)$ contains a proper arithmetic progression P_j of length n_j for all j larger than some initial j_0. Note that $FS(A'_j) \subset [1, 2^j n_j]$, hence the step d_j of the progression P_j cannot exceed 2^j.

This is almost what we need, except that the progressions P_j do not have the same step. This however can be dealt with using the following elementary lemma, which follows from Exercise 3.6.5.

Lemma 12.22 (Coalescence of arithmetic progressions) *[349, 350] Let P_1, P_2 be proper arithmetic progressions of integers of length N_1, N_2 and step $d_1, d_2 > 0$ respectively, where $N_2 \geq 5d_1$ and $N_1 \geq 5d_2$. Then $P_1 + P_2$ contains a proper arithmetic progression of length $N_1 + N_2 - 2$ whose difference is* $\gcd(d_1, d_2)$.

Using this lemma we can see inductively that for j_0 sufficiently large, the set $P_{j_0} + \cdots + P_j$ contains an proper arithmetic progression of length $n_{j_0} + \cdots + n_j - O(j)$ and step $\gcd(d_1, \ldots, d_j)$ for each $j \geq j_0$. The steps $\gcd(d_1, \ldots, d_j)$ are decreasing positive integers and thus must eventually stabilize at some fixed d. Since $P_{j_0} + \cdots + P_j$ is contained in $FS(A'_{j_0}) + \cdots + FS(A'_j)$, which in turn is contained in $FS(A')$, and $n_{j_0} + \cdots + n_j - O(j)$ goes to ∞ as $j \to \infty$, the claim follows. □

The proof of Theorem 12.18 is similar and is left as an exercise.

Exercises

12.5.1 Show that the set A'' used in the proof of Theorem 12.17 obeys (12.2).

12.5.2 [350] Show that there is a constant C such that the following holds. If A is a multiset of positive integers in $[1, n]$ with $|A| \geq Cn$, then $FS(A)$ contains an arithmetic progression of length n. (Hint: Use Theorem 12.10.)

12.5.3 [350] Prove Theorem 12.18. (Hint: use the previous exercise as a substitute for Corollary 12.14.)

12.6 Further applications

In this section, we present a few short applications of Corollary 12.13, taken from [380]. For several applications of Theorem 12.2 we refer to [308, 309] and the references therein.

The following simple lemma will be useful. Let $\mathbf{Z}_n^\times$ denote the residue classes in $\mathbf{Z}_n$ which are coprime to n.