constant cannot contain very large Sidon sets.) What is the analogous statement for the difference constant?

2.2.7 [294] Let p be a prime, let $\theta \in \mathbf{Z}_p \backslash 0$ be a multiplicative generator of $\mathbf{Z}_p$, and let $Z := \mathbf{Z}_{p-1} \times \mathbf{Z}_p$. Let $A \subset Z$ be the set $A := \{(t, \theta^t) : t = 1, \ldots, p-1\}$. Show that A is a Sidon set, and compare this to Exercise 2.2.6. Modify this construction to give an example of a Sidon set $A \subset [0, N]$ for a large integer N such that $|A|$ is comparable to $N^{1/2}$. A similar example can be given by using the discrete parabola $\{(t, t^2) : t \in \mathbf{Z}_p\}$ in $\mathbf{Z}_p \times \mathbf{Z}_p$. For a survey of other constructions of Sidon sets, see [264].

2.2.8 Let N be a large integer. Give examples of finite non-empty sets A, B of integers such that $|A| = |B| = N$ and $\sigma[A], \sigma[B] \leq 2$, but $\sigma[A \cup B] \geq \frac{N}{2}$. This example shows that doubling constants can behave very badly under set union (see however Exercise 2.3.17). On the other hand, establish the inequality $\sigma[A \cup B] \leq \sigma[A] + |B|$; thus adding a *small* set to A will not significantly affect the doubling constant.

2.2.9 Let N be a large integer. Give examples of finite non-empty sets A, B of integers such that $|A| = |B| = N$ and $\sigma[A], \sigma[B] \leq 10$, but $\sigma[A \cap B] \geq \frac{1}{10} N^{1/2}$. (Hint: concatenate a Sidon set with an arithmetic progression.) Compare this result against Exercise 2.2.6. This example shows that doubling constants can behave badly under set intersection (but see Exercise 2.4.7).

2.2.10 Let A be an additive set in Z, and let $\pi : Z \to Z'$ be a group homomorphism. Show by example that $\sigma[\pi(A)]$ is not necessarily less than or equal to $\sigma[A]$. (Hint: this is surprisingly delicate. One way is to start with an additive set C in some additive group Z_0 with $\sigma[C] > \delta[C]$, and consider the additive set $A := ((-C)^n \times \{0\} \times G) \cup (C^n \times X \times \{0\})$ in $Z_0^n \times Z \times G$, where $n \geq 1$ is large, G is a very large finite group, and X is a Sidon set of medium size in a group Z.) See however Exercise 2.3.8 and Exercise 6.5.17.

2.2.11 Let A be an additive set in Z, and let G be a finite subgroup of Z. Show by example that $\sigma[A + G]$ is not necessarily less than or equal to $\sigma[A]$. (Hint: use the previous exercise.)

2.3 Ruzsa distance and additive energy

The doubling constant measures the amount of *internal* additive structure of a single additive set A. We now introduce two useful quantities measuring the amount of common additive structure *between* two additive sets A, B – the Ruzsa distance and the additive energy.

Definition 2.5 (Ruzsa distance) Let A and B be two additive sets with a common ambient group Z. We define the *Ruzsa distance* $d(A, B)$ between these two sets to be the quantity

$$d(A, B) := \log \frac{|A - B|}{|A|^{1/2}|B|^{1/2}}.$$

Thus for instance $d(A, A) = \log \delta[A]$.

We now justify the terminology "Ruzsa distance".

Lemma 2.6 (Ruzsa triangle inequality) *[297] The Ruzsa distance $d(A, B)$ is non-negative, symmetric, and obeys the triangle inequality*

$$d(A, C) \leq d(A, B) + d(B, C)$$

for all additive sets A, B, C with common ambient group Z.

Proof The non-negativity follows from (2.1). The symmetry follows since $B - A = -(A - B)$. Now we prove the triangle inequality, which we can rewrite as

$$|A - C| \leq \frac{|A - B||B - C|}{|B|}.$$

From the identity

$$a - c = (a - b) + (b - c)$$

we see that every element $a - c$ in $A - C$ has at least $|B|$ distinct representations of the form $x + y$ with $(x, y) \in (A - B) \times (B - C)$. The claim then follows. □

For an approximate version of this inequality in which one replaces complete difference sets with nearly complete difference sets (using at least 75% of the differences), see Exercise 2.5.4.

The Ruzsa distance thus satisfies all the axioms of a metric except one; we do not have that $d(A, A) = 0$ for all sets A (also, we have $d(G + x, G + y) = 0$ whenever $G + x, G + y$ are cosets of a group G). Indeed we have a precise characterization on when this Ruzsa distance vanishes:

Proposition 2.7 *Suppose that (A, Z) is an additive set. Then the following are equivalent:*

- $\sigma[A] = 1$ *(i.e. $|A + A| = |A|$);*
- $\delta[A] = 1$ *(i.e. $|A - A| = |A|$, or $d(A, A) = 0$);*
- $d(A, B) = 0$ *for at least one additive set B;*

- $|nA - mA| = |A|$ *for at least one pair of non-negative integers* n, m *with* $n + m \geq 2$*;*
- $|nA - mA| = |A|$ *for all non-negative integers* n, m*;*
- A *is a coset of a finite subgroup* G *of* Z*.*

Proof Apply Proposition 2.2 and the Ruzsa triangle inequality. □

Later on in this chapter we shall generalize this proposition to the case when the Ruzsa distance, difference constant, or doubling constant are a little larger than 0, 0, or 1 respectively, but still fairly small; see Proposition 2.26.

Despite the non-vanishing of the distance $d(A, A)$ in general, it is still a useful heuristic to view the Ruzsa distance as behaving like a metric[1]. Now we relate the difference constant to the doubling constant. From the definition of Ruzsa distance and doubling constant we have the identity

$$d(A, -A) = \log \sigma[A]. \tag{2.4}$$

In particular, from Lemma 2.6 we have

$$\log \delta[A] = d(A, A) \leq 2 \log \sigma[A]$$

and hence we obtain the estimate

$$\delta[A] \leq \sigma[A]^2 \tag{2.5}$$

or in other words that $|A - A| \leq \frac{|A+A|^2}{|A|}$. A similar argument gives the more general estimate

$$|B - B| \leq \frac{|A + B|^2}{|A|} \tag{2.6}$$

for any two additive sets A, B with common ambient group Z.

It turns out that we can conversely bound the doubling constant of a set by its difference constant; see (2.11) below.

Having introduced the Ruzsa distance, we now turn to the closely related notion of *additive energy* $E(A, B)$ between two additive sets.

Definition 2.8 (Additive energy) If A and B are two additive sets with ambient group Z, we define the *additive energy* $E(A, B)$ between A and B to be the quantity

$$E(A, B) := |\{(a, a', b, b') \in A \times A \times B \times B : a + b = a' + b'\}|.$$

[1] One could artificially convert the Ruzsa distance into a genuine metric by identifying A with $A + x$ for all x, and redefining $d(A, A)$ to be zero, or alternatively by introducing the metric space $X := \{A \times \{j\} : A \subseteq Z; 0 < |A| < \infty; j \in \{1, 2\}\}$ – consisting of two copies of each finite non-empty subset of Z (again identifying A with its translations) – with the metric $d_X(A \times \{j\}, B \times \{k\})$ defined to equal $d(A, B)$ if $A \times \{j\} \neq B \times \{k\}$ and equal to 0 otherwise. However there appears to be no significant advantage in working in such an artificial setting.

We observe the trivial bounds

$$|A||B| \leq E(A, B) \leq |A||B| \min(|A||B|). \tag{2.7}$$

The lower bound follows since $a + b = a' + b'$ whenever $(a, b) = (a', b')$. To see the upper bound, observe that if one fixes a, a', b, then $b' = a + a' - b$ is completely determined, and hence $E(A, B) \leq |A|^2|B|$. A similar argument gives $E(A, B) \leq |A||B|^2$. Note that Proposition 2.3 addresses the case when $E(A, B) = |A||B|$.

We will analyze the additive energy more comprehensively in Section 4.2, when we have developed the machinery of Fourier transforms, and in Section 2.5, when we have developed the Balog–Szemerédi–Gowers theorem. For now we concentrate on the elementary properties of this energy. We first observe the symmetry property $E(A, B) = E(B, A)$ and the translation invariance property $E(A + x, B + y) = E(A, B)$ for all $x, y \in Z$. From the trivial observation

$$a + b = a' + b' \iff a - b' = a' - b$$

we also see that $E(A, B) = E(A, -B)$, and similarly if we reflect A to $-A$.

The additive energy reflects the extent to which A intersects with translates of B or $-B$, as the following simple identities show:

Lemma 2.9 *Let A, B be additive sets with ambient group Z. Then we have the identities*

$$|A||B| = \sum_{x \in A+B} |A \cap (x - B)| = \sum_{y \in A-B} |A \cap (B + y)|$$

and

$$\begin{aligned} E(A, B) &= \sum_{x \in A+B} |A \cap (x - B)|^2 \\ &= \sum_{y \in A-B} |A \cap (B + y)|^2 \\ &= \sum_{z \in (A-A) \cap (B-B)} |A \cap (z + A)||B \cap (z + B)|. \end{aligned}$$

In particular, if we let $r_{A+B}(n)$ denote the number of representations of n as $a + b$ for some $a \in A$ and $b \in B$, and define $r_{A-B}(n)$ similarly, then we have

$$|A||B| = \sum_n r_{A+B}(n) = \sum_n r_{A-B}(n); \; E(A, B) = \sum_n r_{A+B}(n)^2 = \sum_n r_{A-B}(n)^2.$$

Proof A simple counting argument yields

$$|A||B| = \sum_{x \in A+B} |\{(a, b) \in A \times B : a + b = x\}| = \sum_{x \in A+B} |A \cap (x - B)|;$$

By replacing B with $-B$ we similarly obtain $|A||B| = \sum_{y \in A-B} |A \cap (B+y)|$. This gives the first set of identities. For the second set we compute

$$\begin{aligned}
&\sum_{x \in A+B} |A \cap (x-B)|^2 \\
&\quad = \sum_{x \in A+B} |\{(a,b) \in A \times B : a+b = x\}|^2 \\
&\quad = \sum_{x \in A+B} |\{(a,a',b,b') \in A \times A \times B \times B : a+b = a'+b' = x\}| \\
&\quad = |\{(a,a',b,b') \in A \times A \times B \times B : a+b = a'+b'\}| \\
&\quad = |\{(a,a',b,b') \in A \times A \times B \times B : a-b' = a'-b\}| \\
&\quad = \sum_{y \in A-B} |\{(a,b') \in A \times B : a-b' = a'-b\}|^2 \\
&\quad = \sum_{y \in A-B} |A \cap (B+y)|^2
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{z \in (A-A)\cap(B-B)} |A \cap (z+A)||B \cap (z+B)| \\
&\quad = \sum_{z \in (A-A)\cap(B-B)} |\{(a,a',b,b') \in A \times A \times B \times B : z = a-a' = b'-b\}| \\
&\quad = |\{(a,a',b,b') \in A \times A \times B \times B : a-a' = b'-b\}| \\
&\quad = |\{(a,a',b,b') \in A \times A \times B \times B : a+b = a'+b'\}|
\end{aligned}$$

and the claims follow from the definition of $E(A, B)$. The last identity follows since $r_{A+B}(n) = |A \cap (n-B)|$ and $r_{A-B}(n) = |A \cap (B+n)|$. □

As a consequence of this Lemma we have the following inequalities, which assert that pairs of sets with small Ruzsa distance have large additive energy, and pairs with large additive energy have large intersection (after translating and possibly reflecting one of the sets).

Corollary 2.10 *Let A, B be additive sets. Then there exists $x \in A+B$ and $y \in A-B$ such that*

$$|A \cap (x-B)|, |A \cap (B+y)| \geq \frac{E(A,B)}{|A||B|} \geq \frac{|A||B|}{|A \mp B|} \tag{2.8}$$

for either choice of sign $\pm$. In particular all of the above quantities are bounded by $|(A-A) \cap (B-B)|$. Finally we have the Cauchy–Schwarz inequality

$$E(A,B) \leq E(A,A)^{1/2} E(B,B)^{1/2}. \tag{2.9}$$

Proof From Lemma 2.9 and Cauchy–Schwarz we have

$$\frac{E(A,B)}{|A||B|} \geq \frac{|A||B|}{|A \pm B|}.$$

Also, from the last part of Lemma 2.9 we have

$$E(A,B) \leq |A||B| \max_{x \in A+B} r_{A+B}(x), |A||B| \max_{y \in A-B} r_{A-B}(y)$$

which establishes (2.8). To bound $|A \cap (x - B)|$ and $|A \cap (B + y)|$, observe that if $z \in A \cap (x - B)$, then $A \cap (x - B) \subset z + ((A - A) \cap (B - B))$, hence $|A \cap (x - B)| \leq |(A - A) \cap (B - B)|$, and similarly $|A \cap (B + y)| \leq |(A - A) \cap (B - B)|$. Finally, (2.9) follows from the formula $E(A,B) = \sum_{z \in (A-A)\cap(B-B)} |A \cap (z + A)||B \cap (z + B)|$ from Lemma 2.9 and the Cauchy–Schwarz inequality. □

Another connection in a similar spirit is

Lemma 2.11 *Let A, B be additive sets. Then for any $x \in A + B$ we have $|A \cap (x - B)| \leq \frac{|A-B|^2}{|A+B|}$.*

Proof (Lev Vsevolod, private communication) We can rewrite the inequality as

$$|\{(a,b,c) \in A \times B \times (A+B) : a + b = x\}| \leq |(A - B) \times (A - B)|.$$

Now for each (a, b, c) in the set on the left-hand side, we can write $c = a_c + b_c$ for some $a_c \in A, b_c \in B$, and then form the pair $(a - b_c, a_c - b) \in (A - B) \times (A - B)$. Using the identity $c = x - (a - b_c) + (a_c - b)$ we can verify that this map is injective. The claim follows. □

Corollary 2.12 *Let A, B be additive sets with ambient group Z. Then there exists $x \in A + B$ such that*

$$\frac{|A - B|^2}{|A \cap (x - B)|} \leq \frac{|A - B|^2 |A||B|}{E(A,B)} \leq \frac{|A - B|^3}{|A||B|}. \tag{2.10}$$

Furthermore we have

$$d(A, -B) \leq 3d(A, B).$$

Proof The inequalities in (2.10) follow from (2.8), and the final inequality $d(A, -B) \leq 3d(A, B)$ then follows from Lemma 2.11 and the definition of Ruzsa distance. □

From (2.10) and and (2.5) we obtain the inequalities

$$\delta[A]^{1/2} \leq \sigma[A] \leq \delta[A]^3 \tag{2.11}$$

which were first observed in [289]. Thus an additive set has small doubling constant if and only if its difference constant is small. It is not known whether the lower bound is best possible. However, the upper bound can be improved to $\sigma[A] \leq \delta[A]^2$ using Plünnecke inequalities; see Exercise 6.5.15.

We now show how the Ruzsa distance can be used to control iterated sum sets. We begin with a lemma which controls iterated sum sets of "most" of $A + B$.

Lemma 2.13 *Let A and B be additive sets in a common ambient group. Then there exists $S \subset A + B$ such that*

$$|\{(a, b) \in A \times B : a + b \in S\}| \geq |A||B|/2 \tag{2.12}$$

and such that

$$|A + B + nS| \leq \frac{2^n |A + B|^{2n+1}}{|A|^n |B|^n} \tag{2.13}$$

for all integers $n \geq 0$.

Note that (2.12) gives a lower bound on $|S|$, namely

$$|S| \geq \max(|A|, |B|)/2. \tag{2.14}$$

Proof If we define S to be the set of all $x \in A + B$ such that

$$|\{(a, b) \in A \times B : a + b = x\}| \geq \frac{|A||B|}{2|A + B|}$$

then we have

$$|\{(a, b) \in A \times B : a + b \in (A + B)\backslash S\}| < |A + B| \frac{|A||B|}{2|A + B|}$$

which gives (2.12).

Now we prove (2.13). A typical element of $A + B + nS$ can be written as

$$a_0 + s_1 + s_2 + \cdots + s_n + b_{n+1}$$

where $a_0 \in A$, $b_{n+1} \in B$, and $s_1, \ldots, s_n \in S$. By definition of S, we can expand this in at least $(\frac{|A||B|}{2|A+B|})^n$ different ways as

$$a_0 + (b_1 + a_1) + (b_2 + a_2) + \cdots + (b_n + a_n) + b_{n+1}$$

where $b_i \in B$, $a_i \in A$, and $b_i + a_i = s_i$ for all $1 \leq i \leq n$. We regroup this as the sum of $n + 1$ elements from $A + B$,

$$(a_0 + b_1) + (a_1 + b_2) + \cdots + (a_n + b_{n+1})$$

and observe that for fixed $a_0, s_1, \ldots, s_n, b_{n+1}$, the quantities $a_0 + b_1, a_1 + b_2, \ldots, a_n + b_{n+1}$ completely determine all the variables $a_0, \ldots, a_n, b_1, \ldots, b_{n+1}$. Thus we have shown that every element of $A + B + nS$ has at least $(\frac{|A||B|}{2|A+B|})^n$ representations of the form $t_0 + \cdots + t_n$ where each $t_i \in A + B$. The claim then follows. □

This result can then be used, together with the Ruzsa triangle inequality, to deduce control on iterated sum sets of A and B; see Exercise 2.3.10. However we will pursue an approach that gives slightly better bounds in the next section (and an even better result will be developed in Section 6.5).

Exercises

2.3.1 If $\phi : Z' \to Z$ is a surjective group homomorphism whose kernel $\ker(\phi) := \phi^{-1}(\{0\})$ is finite, and A, B are additive sets in Z, show that $d(\phi^{-1}(A), \phi^{-1}(B)) = d(A, B)$. Also show that $d(A + x, B + y) = d(A, B)$ for any $x, y \in Z$.

2.3.2 If A, B, C, D are additive sets in Z, show that

$$d(A, B) - \frac{1}{2}\log|C||D| \leq d(A + C, B + D) \leq d(A, B) + \log|C - D|$$

and

$$d(A, B \cup C) \leq \max(d(A, B), d(A, C)) + \frac{1}{2}\log 2.$$

If A', B' are additive sets in Z', show that

$$d(A \times A', B \times B') = d(A, B) + d(A', B').$$

2.3.3 Let A, B be additive sets with common ambient group. Show that $d(A, B) \leq \frac{1}{2}\log|A| + \frac{1}{2}\log|B|$, and that $d(A, B) = \frac{1}{2}\log|A| + \frac{1}{2}\log|B|$ if and only if $d(A, -B) = \frac{1}{2}\log|A| + \frac{1}{2}\log|B|$.

2.3.4 Let A, B, C be additive sets in Z. Show that

$$d(A, C) \leq d(A, B) + \frac{1}{2}\log\frac{|B|}{|C|} \tag{2.15}$$

whenever $C \subseteq B$; this shows that the Ruzsa distance $d(A, B)$ is stable under refinement of one or both of the sets A, B. By combining this inequality with the triangle inequality $d(A, -B) \leq d(A, (x - A) \cap B) + d((x - A) \cap B, -B)$, give another proof of Lemma 2.11.

2.3.5 Show that for any $n \geq 1$, there exists an additive set A such that $|A| = 4^n$, $|A + A| = 10^n$, and $|2A - A| = 28^n$. Thus it is not possible to obtain an estimate of the form $|2A - A| = O(\sigma^2[A]|A|)$.

2.3.6 Let A, B be additive sets with common ambient group. Show that $e^{-2d(A,B)}|A| \leq |B| \leq e^{2d(A,B)}|A|$. Thus sets which are close in the Ruzsa distance are necessarily close in cardinality also. Of course the converse is far from true.

2.3.7 Let A, B be additive sets with common ambient group Z. Show that $d(A, B) = 0$ if and only if A, B are cosets of the same finite subgroup G of Z. (We shall generalize this result later; see Proposition 2.27.)

2.3.8 Let A be an additive set in an additive group Z, and let G be a finite subgroup of Z. Show that $\sigma[A+G] \leq \frac{|3A|}{|A|}$. (Hint: apply the Ruzsa triangle inequality to $2A$, $-A$, and G.) Conclude that if $\pi : Z \to Z'$ is a group homomorphism then $\sigma[\pi(A)] \leq \frac{|3A|}{|A|}$. One cannot replace the tripling constant $\frac{|3A|}{|A|}$ with the doubling constant; see Exercise 2.2.10. See however Exercise 6.5.17.

2.3.9 Let K be a large integer, and let $A = B = \{e_1, \ldots, e_K\}$ be the standard basis of $\mathbf{Z}^K$. Show that if S is any subset of $A+B$ obeying (2.12) then

$$|A+B+nS| = \Omega_n\left(\frac{|A+B|^{2n+1}}{|A|^n|B|^n}\right)$$

where we are using the Landau notation $\Omega()$. This shows that Lemma 2.13 cannot be significantly improved (except possibly by improving the bound (2.14)).

2.3.10 Let A, B be additive sets with common ambient group such that $|A+B| \leq K|A|^{1/2}|B|^{1/2}$ for some $K \geq 1$. Using Lemma 2.13 and many applications of the Ruzsa triangle inequality, establish the estimate

$$|n_1A - n_2A + n_3B - n_4B| = O_{n_1,n_2,n_3,n_4}\left(K^{O_{n_1,n_2,n_3,n_4}(1)}|A|^{1/2}|B|^{1/2}\right)$$

for all integers n_1, n_2, n_3, n_4. In particular, establish the bounds

$$\begin{aligned} &d(n_1A - n_2A + n_3B - n_4B, n_5A - n_6A + n_7B - n_8B) \\ &\quad \leq O_{n_1,\ldots,n_8}(1 + d(A, B)) \end{aligned}$$

for all integers $n_1, \ldots, n_8$. We shall improve this bound slightly in Corollary 2.23 and Corollary 2.24; see also Corollary 2.19 for the "tensor power trick" that can eliminate lower order terms such as the implicit constant preceding the $K^{O_{n_1,n_2,n_3,n_4}(1)}$ factor.

2.3.11 Let G and H be subgroups of Z. Show that

$$d(G, H) = \log\frac{|G|^{1/2}|H|^{1/2}}{|G \cap H|}$$

Conclude that $d(G,H) = d(G, G+H) + d(G+H, H) = d(G, G \cap H) + d(G \cap H, H)$. Also, if K is another subgroup of Z, prove the contractivity properties $d(G+K, H+K) \leq d(G,H)$ and $d(G \cap K, H \cap K) \leq d(G,H)$. Note that the Ruzsa distance, when restricted to subgroups of Z, is indeed a genuine metric, thanks to Proposition 2.7. See also Exercises 2.4.7 and 2.4.8 below.

2.3.12 Let A be an additive set. Show that

$$\sigma[A \cup (-A)] \leq 2\sigma[A] + \sigma[A]^2.$$

Thus a set with small doubling can be embedded in a symmetric set (i.e. a set B such that $-B = B$) with small doubling which has at most twice the cardinality.

2.3.13 [289] Let A be an additive set. Prove the inequalities $|A - A| \leq |A + A|^{3/2}$ and $|A + A| \leq |A - A|^{3/2}$. (Hint: use (2.11), Corollary 2.12 and (2.1).)

2.3.14 [26] Let A be an additive set. Show that there exists an element $x \in A - A$ such that the set $F := A \cap (x + A)$ has size $|F| \geq |A|/\sigma[A]$ and doubling constant $\sigma[F] \leq \sigma[A]^2$. Thus every additive set A of small doubling contains a large symmetric subset F of small doubling, though the set F may be symmetric around a non-zero origin $x/2$.

2.3.15 Let A, B be additive sets with common ambient group Z. Show that $\delta[A] \leq e^{2d(A,B)}$ and $\sigma[A] \leq e^{6d(A,B)}$. Thus only sets with small doubling constant can be close to other sets in the Ruzsa metric. (The 6 can be lowered to a 4, see Exercise 6.5.15.)

2.3.16 Let A, B be additive sets with common ambient group Z. Show that $\sigma[A \cup B] \leq e^{d(A,B)} + 2e^{4d(A,B)}$. Thus a pair of sets which are close in the Ruzsa metric can be embedded in a slightly larger set with small doubling. In the converse direction, establish the estimate

$$d(A, B) \leq \log \sigma[A \cup B] + \frac{1}{2} \log \frac{|A \cup B|}{|A|} + \frac{1}{2} \log \frac{|A \cup B|}{|B|}.$$

2.3.17 Let A, B be additive sets with common ambient group Z, such that $\sigma[A], \sigma[B] \leq K$ for some $K \geq 1$, and such that $A \cap B$ is non-empty. Show that

$$\sigma[A \cup B] \leq 2K + K^3 \frac{\min(|A|, |B|)}{|A \cap B|}.$$

Thus the union of sets with small doubling remains small doubling provided that those two sets had substantial intersection.

2.3.18 [40], [41] Let $K \geq 1$, and let A_1, A_2, A_3 be additive sets with common ambient group Z, such that

$$|A_j \cap A_3| \geq \frac{1}{K}|A_j| \text{ and } |A_j + A_j| \leq K|A_j|$$

for all $j = 1, 2, 3$. Prove that $|A_1 + A_2| \leq K^6 |A_3|$. Hint: use the triangle inequality

$$\begin{aligned} d(A_1, -A_2) \leq\ & d(A_1, -(A_1 \cap A_3)) + d(-(A_1 \cap A_3), A_2 \cap A_3) \\ & + d(A_2 \cap A_3, -A_2) \end{aligned}$$