Similarly for permutations. Conclude from this and the preceding exercise that

$$d(G_1, G_2) \leq d(G_1 + G_3, G_2 + G_3) + 2 \log K_1 K_2 K_3$$

and compare this with Exercise 2.3.11. (A corresponding statement exists for intersections but is somewhat tricky to establish.)

2.4.9 For any integers $K, n_1, n_2 \geq 1$, give an example of an additive set A with $\sigma[A] = K$ and $\sigma[n_1 A - n_2 A] = \Omega_{n_1,n_2}(K^{n_1+n_2})$.

2.4.10 Let A, B be additive sets in a common ambient group Z. Show that $\sigma[A + B] \leq (\sigma[A]\sigma[B])^C$ where $C \geq 1$ is an absolute constant. (Hint: use Proposition 2.26 to place A and B inside translates of approximate groups. To obtain lower bounds on $|A + B|$, use the inequality

$$|A + B| \geq \frac{|A||B|}{|(A - A) \cap (B - B)|}$$

from (2.8).)

2.4.11 Prove Proposition 2.4.11. (Hint: to construct the approximate group H, one possible choice is $H = A - A + B - B$.)

2.4.12 Try to improve upon the constant 5 in (2.17), by using the Ruzsa triangle inequality instead of the Ruzsa covering lemma. This exercise demonstrates that the triangle inequality is slightly sharper than the covering lemma when one wants cardinality bounds, but the covering lemmas of course give much more information than just cardinality.

2.4.13 [209] Let A, B be additive sets in an ambient group Z, and let G be the group generated by A. Show that there exists an additive set $B' \Subset B$ such that B' is contained in a coset of G, and such that $|A + B'| \leq \frac{|B'|}{|B|}|A + B|$.

2.4.14 Let A, B, A', B' be additive sets with common ambient group Z. Establish the inequality $d(A + A', B + B') = O(d(A, B) + d(A', B'))$. (Hint: argue as in Exercise 2.4.10.) Conclude that if $\phi : Z \to Z'$ is a group homomorphism, then $d(\phi(A), \phi(B)) = O(d(A, B))$. Thus group homomorphisms are "Lipschitz" with respect to the Ruzsa distance.

2.5 The Balog–Szemerédi–Gowers theorem

In the previous sections we have only considered complete sum sets $A + B$ and complete difference sets $A - B$. In many applications one only controls a partial collection of sums and differences. Fortunately, there is a very useful tool, the *Balog–Szemerédi–Gowers theorem*, which allows one to pass from control of partial sum and difference sets to control of complete sum and difference sets (after refining the sets slightly). We begin with some notation.

Definition 2.28 (Partial sum sets) If A, B are additive sets with common ambient group Z, and G is a subset of $A \times B$, we define the *partial sum set*

$$A \overset{G}{+} B := \{a + b : (a, b) \in G\}$$

and the *partial difference set*

$$A \overset{G}{-} B := \{a - b : (a, b) \in G\}.$$

One may like to think of G as a bipartite graph connecting A and B. Note that when $G = A \times B$ is complete, then the notion of partial sum set and partial difference set collapse to just the complete sum set and difference set.

Partial sum sets and partial difference sets are not as nice to work with algebraically as complete sum sets. In particular, the above machinery of sum set estimates do not directly yield any conclusion if one only assumes that the cardinality $|A \overset{G}{+} B|$ of a partial sum set is small. Note that even when G is very large, it is possible for $|A \overset{G}{+} B|$ to be small while $|A + B|$ is large; see exercises. Fortunately, the Balog–Szemerédi–Gowers theorem, which we will present shortly, does allow us to conclude information on complete sum sets from information on partial sum sets, if we are willing to refine A and B by a small factor (i.e. replace A and B by subsets A' and B' which are only slightly smaller than A and B).

The first result in this direction was by Balog and Szemerédi [16], using the regularity lemma. A different, more effective proof, was found by Gowers [137] (with a slight refinement by Bourgain [38]), in particular with dependence of constants that are only polynomial in nature. Here we present a modern formulation of the theorem, following [340].

Theorem 2.29 (Balog–Szemerédi–Gowers theorem) *Let A, B be additive sets in an ambient group Z, and let $G \subseteq A \times B$ be such that*

$$|G| \geq |A||B|/K \text{ and } |A \overset{G}{+} B| \leq K'|A|^{1/2}|B|^{1/2}$$

for some $K \geq 1$ and $K' > 0$. Then there exists subsets $A' \subseteq A$, $B' \subseteq B$ such that

$$|A'| \geq \frac{|A|}{4\sqrt{2}K} \tag{2.18}$$

$$|B'| \geq \frac{|B|}{4K} \tag{2.19}$$

$$|A' + B'| \leq 2^{12} K^4 (K')^3 |A|^{1/2}|B|^{1/2}. \tag{2.20}$$

In particular we have

$$d(A', -B') \leq 5 \log K + 3 \log K' + O(1).$$

The proof of this theorem is graph-theoretical. It is elementary, but a little lengthy and so we postpone it to Section 6.4. One can of course combine this theorem with Corollary 2.24 and Proposition 2.26 to gain more information on the iterated sum and difference sets of A'' and B''. It is likely that the factor of $2^{12}K^4(K')^3$ in (2.20) can be improved. However, the bounds (2.18), (2.19) cannot be significantly improved; see exercises.

To apply the Balog–Szemerédi–Gowers theorem, it is convenient to introduce the following lemma connecting large additive energy to small partial sum sets or small partial difference sets.

Lemma 2.30 *Let A, B be additive sets in an ambient group Z, and let G be a non-empty subset of $A \times B$. Then*

$$E(A, B) \geq \frac{|G|^2}{|A \overset{G}{+} B|}, \frac{|G|^2}{|A \overset{G}{-} B|}.$$

Conversely, if $E(A, B) \geq |A|^{3/2}|B|^{3/2}/K$ for some $K \geq 1$, then there exists $G \subseteq A \times B$ such that

$$|G| \geq |A||B|/2K; \qquad |A \overset{G}{+} B| \leq 2K|A|^{1/2}|B|^{1/2}.$$

and similarly there exists $H \subseteq A \times B$ such that

$$|H| \geq |A||B|/2K; \qquad |A \overset{H}{-} B| \leq 2K|A|^{1/2}|B|^{1/2}.$$

Proof Observe that

$$\sum_{x \in A \overset{G}{+} B} |\{(a, b) \in G : a + b = x\}| = |G|$$

and hence by Cauchy–Schwarz

$$\frac{\sum_{x \in A \overset{G}{+} B} |\{(a, b) \in G : a + b = x\}|^2 \geq |G|^2}{|A \overset{G}{+} B|}.$$

But the left-hand side is equal to

$$|\{(a, a', b, b') \in A \times A \times B \times B : a + b = a' + b'; (a, b), (a', b') \in G\}|$$

which was less than $E(A, B)$. This proves that $E(A, B) \geq |G|^2/|A \overset{G}{+} B|$; using the symmetry $E(A, B) = E(A, -B)$ we thus also obtain $E(A, B) \geq |G|^2/|A \overset{G}{-} B|$.

Now assume $E(A, B) \geq |A|^{3/2}|B|^{3/2}/K$. Then by Lemma 2.9 we have

$$\sum_{x \in A+B} |A \cap (x - B)|^2 \geq \frac{|A|^{3/2}|B|^{3/2}}{K}.$$

If we set $S := \{x \in A + B : |A \cap (x - B)| \geq |A|^{1/2}|B|^{1/2}/2K\}$, we then have (by Lemma 2.9 again)

$$\sum_{x \in S} |A \cap (x - B)|^2 \geq \frac{|A|^{3/2}|B|^{3/2}}{K} - \frac{|A||B||A|^{1/2}|B|^{1/2}}{2K} = \frac{|A|^{3/2}|B|^{3/2}}{2K}.$$

Now observe from Lemma 2.9 again that

$$\frac{|S||A|^{1/2}|B|^{1/2}}{2K} \leq \sum_{x \in S} |A \cap (x - B)| \leq |A||B|$$

and hence

$$|S| \leq 2K|A|^{1/2}|B|^{1/2}.$$

Now let $G := \{(a, b) \in A \times B : a + b \in S\}$, then clearly $A \overset{G}{+} B \subseteq S$ and hence

$$|A \overset{G}{+} B| \leq 2K|A|^{1/2}|B|^{1/2}.$$

Furthermore we have

$$\begin{aligned} |G| &= \sum_{x \in S} |\{(a, b) \in A \times B : a + b = x\}| \\ &= \sum_{x \in S} |A \cap (x - B)| \\ &\geq \sum_{x \in S} \frac{|A \cap (x - B)|^2}{|A|^{1/2}|x - B|^{1/2}} \\ &\geq \frac{|A|^{3/2}|B|^{3/2}/2K}{|A|^{1/2}|B|^{1/2}} \\ &= |A||B|/2K. \end{aligned}$$

This gives the desired set G. The construction of H follows by using the symmetry $E(A, B) = E(A, -B)$. □

Combining this Lemma with the Balog–Szemerédi–Gowers theorem, we can obtain a characterization of pairs of sets with large additive energy.

Theorem 2.31 (Balog–Szemerédi–Gowers theorem, alternative version) *Let A, B be additive sets in an ambient group Z, and let $K \geq 1$. Then the following statements are equivalent up to constants, in the sense that if the jth property holds for some absolute constant C_j, then the kth property will also hold for some absolute constant C_k depending on C_j:*

(i) $E(A, B) \geq K^{-C_1}|A|^{3/2}|B|^{3/2}$;

(ii) there exists $G \subset A \times B$ *such that* $|G| \geq K^{-C_2}|A||B|$ *and* $|A \overset{G}{+} B| \leq K^{C_2}|A|^{1/2}|B|^{1/2}$;

(iii) there exists $G \subset A \times B$ *such that* $|G| \geq K^{-C_3}|A||B|$ *and* $|A \overset{G}{-} B| \leq K^{C_3}|A|^{1/2}|B|^{1/2}$*;*

(iv) there exists subsets $A' \subseteq A$, $B' \subseteq B$ *with* $|A'| \geq K^{-C_4}|A|$, $|B'| \geq K^{-C_4}|B|$, *and* $d(A', B') \leq C_4 \log K$*;*

(v) there exists a K^{C_5}*-approximate group* H *and* $x, y \in Z$ *such that* $|A \cap (H + x)|, |B \cap (H + y)| \geq K^{-C_5}|H|$ *and* $|A|, |B| \leq K^{C_5}|H|$.

We leave the proof of this theorem to the exercises. Theorem 2.31 should be compared with Exercise 2.3.22, which is the $K = 1$ case of this Theorem. As with Proposition 2.27, this Theorem is restricted to sets A, B which are close in cardinality (see exercises). We shall address the question of sets A, B of widely differing cardinalities in the next section.

Exercises

2.5.1 Let A, B be additive sets with common ambient group Z such that $E(A, B) \geq K^{-1}|A|^{3/2}|B|^{3/2}$. Show that $K^{-2}|A| \leq |B| \leq K^2|A|$, and show by means of an example that these bounds cannot be improved.

2.5.2 Give an example of an additive set $A \subset \mathbf{Z}$ of cardinality N, and a set $G \subset A \times A$ of cardinality $N^2/4$, such that $|A \overset{G}{+} A| \leq N$ but $|A + A| \geq N^2/8$. (Hint: concatenate a Sidon set with an arithmetic progression.)

2.5.3 Let $N \gg K \gg 1$ be large integers, with N a multiple of K. Give an example of sets $A, B \subset \mathbf{Z}$ of cardinality $|A| = |B| = N$ and a subset $G \subset A \times B$ of cardinality $|G| = |A||B|/K$ with the property that $|A \overset{G}{+} B| \leq 2N$, but such that $|A'' + B''| \geq N^2/K^2$ whenever $A'' \subset A$ and $B'' \subset B$ is such that $|A''| \geq 2|A|/K$. (Hint: take B to be a long progression, and take A to be a short progression concatenated with some generic integers.) This shows that the conditions (2.18), (2.19) in Theorem 2.29 cannot be significantly improved.

2.5.4 Let A, B, C be additive sets in an ambient group Z, let $0 < \varepsilon < 1/4$, and let $G \subset A \times B$, $H \subset B \times C$ be such that $|G| \geq (1 - \varepsilon)|A||B|$ and $|H| \geq (1 - \varepsilon)|B||C|$. Show that there exists subsets $A' \subseteq A$ and $C' \subseteq C$ with $|A'| \geq (1 - \varepsilon^{1/2})|A|$ and $|C'| \geq (1 - \varepsilon^{1/2})|C|$ such that $|A' - C'| \leq |A \overset{G}{-} B||B \overset{H}{-} C|/(1 - 2\varepsilon^{1/2})|B|$. (Hint: show that at most $\varepsilon^{1/2}|B|$ elements of B have a G-degree of less than $(1 - \varepsilon^{1/2})|A|$, and similarly at most $\varepsilon^{1/2}|B|$ elements have a H-degree of less than $(1 - \varepsilon^{1/2})|C|$.) This result is can be used as a substitute for the Balog–Szemerédi–Gowers theorem in the case when the graph G is extremely dense; it has the advantage that it does not require A, B, C to be comparable in size and