it does not lose any constants in the limit $\varepsilon \to 0$; indeed it collapses to Ruzsa's triangle inequality in that limit.

2.5.5 Prove Theorem 2.31. (Hint: for K large, e.g. $K \geq 1.1$, one can use the Balog–Szemerédi–Gowers theorem and Proposition 2.27. For K small, e.g. $1 \leq K < 1.1$, one can use Exercise 2.5.4 as a substitute for the Balog–Szemerédi–Gowers theorem.)

2.5.6 [80] Let A, B be additive sets with common ambient group such that $|A| = |B| = N$ and $|A + A| \leq KN$. Suppose also that $|A \stackrel{G}{+} B| \leq KN$, where $G \subset A \times B$ is a bipartite graph such that every element of B is connected to at least $K^{-1}N$ elements of A. Show that $|A + B| \leq K^{O(1)}N$ and $|B + B| \leq K^{O(1)}N$. (Hint: write the elements of $A + B$ in the form $x - y + z$ where $x \in A + A$, $y \in A + A$, and $z \in A \stackrel{G}{+} B$.)

2.5.7 [80] Let A be an additive set such that $|A \stackrel{G}{+} A| \leq K|A|$, where $G \subset A \times A$ is such that every element of A is connected via G to at least $K^{-1}|A|$ elements of A. Show that one can partition A into $O(K^{O(1)})$ subsets $A_1, \ldots, A_m$ such that $|A_i + A_i| = O(K^{O(1)}|A|)$ for each $1 \leq i \leq m$. (Hint: use the Balog–Szemerédi–Gowers theorem and an iteration argument to obtain most of the subsets, and then Exercise 2.5.6 to deal with the remainder.)

2.6 Symmetry sets and imbalanced partial sum sets

The Balog–Szemerédi–Gowers theorem is a very powerful tool when studying two additive sets A, B with additive energy $E(A, B)$ close to $|A|^{3/2}|B|^{3/2}$; however from (2.7) we see that this situation only occurs when $|A|$ and $|B|$ are comparable in size. This leaves open the question of what happens in the case $|A| \gg |B|$ (say) and $E(A, B)$ is close to the upper bound of $|A||B|^2$ given by (2.7). A special sub-case of this (thanks to (2.8)) is the case when $|A + B|$ or $|A - B|$ is comparable to $|A|$. Note that Proposition 2.2 already gives an answer to this question in the extreme case when $|A + B| = |A|$ or $|A - B| = |A|$ (or equivalently if $E(A, B) = |A||B|^2$; see Exercise 2.3.22). However, an example of Ruzsa [297] shows that things become bad when $|A|$ and $|B|$ are very widely separated; see the exercises.

If however we are prepared to endure logarithmic-type losses in the ratio $|A|/|B|$ (or more precisely losses of the form $(|A|/|B|)^\varepsilon$ where ε can be chosen to be small), then one can recover a reasonable theory. In analogy with Proposition 2.2, one expects that if $|A + B|$ is comparable to $|A|$, or if $E(A, B)$ is close to $|A||B|^2$, then there should be an approximate group H such that A is approximately the union of translates of H, and B is approximately contained in a single translate of H. To achieve this will be the main objective of this section.

In the extreme case when $|A+B| = |A|$ or $E(A, B) = |A||B|^2$, the approximate group H was in fact an exact group and in the proof of Proposition 2.2 it was constructed as the symmetry group $\mathrm{Sym}_1(A)$ of the larger additive set A. In the general case this symmetry group is likely to be trivial. However, a more general notion is still useful.

Definition 2.32 (Symmetry sets) Let (A, Z) be an additive set. For any non-negative real number $\alpha \geq 0$, define the *symmetry set* $\mathrm{Sym}_\alpha(A) \subseteq Z$ *at threshold* α to be the set

$$\mathrm{Sym}_\alpha(A) := \{h \in Z : |A \cap (A+h)| \geq \alpha |A|\}.$$

Note that $\mathrm{Sym}_1(A) = \{h \in Z : A + h = A\}$ is the same symmetry group applied in the proof of Proposition 2.2. The other symmetry sets are not groups in general, but nevertheless they are still symmetric (so $-\mathrm{Sym}_\alpha(A) = \mathrm{Sym}_\alpha(A)$) and contain the origin, and they obey the nesting property $\mathrm{Sym}_\alpha(A) \subseteq \mathrm{Sym}_\beta(A)$ for $\alpha \geq \beta$. It is also clear that $\mathrm{Sym}_\alpha(A) \subseteq A - A$ for all $0 < \alpha \leq 1$. Note that as $\mathrm{Sym}_\alpha(A)$ is empty for $\alpha > 1$ and equal to all of Z for $\alpha \leq 0$, we shall mostly restrict ourselves to the non-trivial region where $0 < \alpha \leq 1$.

We now relate the size of these symmetry sets to the additive energy. From Lemma 2.9 we have

$$E(A, A) = \sum_{h \in A - A} |A \cap (A+h)|^2$$

and hence for any $0 < \alpha \leq 1$ and the crude bounds $|A \cap (A+h)| \leq |A|$ when $h \in \mathrm{Sym}_\alpha(A)$ and $|A \cap (A+h)| \leq \alpha|A|$ when $h \notin \mathrm{Sym}_\alpha(A)$, we have

$$\alpha^2 |A|^2 |\mathrm{Sym}_\alpha(A)| \leq E(A, A) \leq \alpha^2 |A|^2 |A - A| + |A|^2 |\mathrm{Sym}_\alpha(A)|,$$

which indicates that $\mathrm{Sym}_\alpha(A)$ should be large whenever the energy is large. In particular, from (2.7) we have

$$|\mathrm{Sym}_\alpha(A)| \leq |A|/\alpha^2. \tag{2.21}$$

Now let A, B be additive sets in an additive group Z. From Lemma 2.9 again, we have

$$E(A, B) = \sum_{b, b' \in B} |A \cap (A + b - b')|$$

and hence for any $0 < \alpha \leq 1$ we have

$$E(A, B) \leq |B|^2 \alpha |A| + |A| |\{(b, b') \in B : b - b' \in \mathrm{Sym}_\alpha(A)\}|.$$

In particular, if $E(A, B) \geq 2\alpha|A||B|^2$, then we conclude that there is a set $G \subset B \times B$ of cardinality $|G| \geq \alpha|B|^2$ such that

$$B \stackrel{G}{-} B \subseteq \mathrm{Sym}_\alpha(A). \tag{2.22}$$

At first glance it seems that one may now be able to apply the symmetric Balog–Szemerédi–Gowers theorem. However, the fact that A is much larger than B means that $B \stackrel{G}{-} B$ may be much larger than B (compare (2.22) to (2.21)). To get around this difficulty we need to iterate this construction, and exploit the fact that $\mathrm{Sym}_\alpha(A)$ behaves like a group. This is already clear when $\alpha = 1$, when $\mathrm{Sym}_1(A)$ is indeed a genuine group; the following lemma shows that this behavior persists in an approximate sense for α less than 1.

Lemma 2.33 *Let A be an additive set. Then we have*

$$\mathrm{Sym}_{1-\varepsilon}(A) + \mathrm{Sym}_{1-\varepsilon'}(A) \subseteq \mathrm{Sym}_{1-\varepsilon-\varepsilon'}(A) \tag{2.23}$$

whenever $\varepsilon, \varepsilon' > 0$. Furthermore, if $0 < \alpha \leq 1$ and $S \subseteq \mathrm{Sym}_\alpha(A)$ is a non-empty set, then there exists a set $G \subseteq |S|^2$ with

$$|G| \geq \alpha^2|S|^2/2 \tag{2.24}$$

such that

$$S \stackrel{G}{-} S \subseteq \mathrm{Sym}_{\alpha^2/2}(A). \tag{2.25}$$

Proof To verify the first claim, observe that if $x \in \mathrm{Sym}_{1-\varepsilon}(A)$ and $y \in \mathrm{Sym}_{1-\varepsilon'}(A)$ then

$$|(A + x)\backslash A| = |A| - |A \cap (A + x)| \leq \varepsilon|A|$$

and

$$|(A + x)\backslash(A + x + y)| = |A| - |A \cap (A + y)| \leq \varepsilon'|A|,$$

and hence

$$|A \cap (A + x + y)| \geq |(A + x) \cap A \cap (A + x + y)| \geq (1 - \varepsilon - \varepsilon')|A|$$

which proves (2.23).

Now we prove the second claim. By definition of S, we see that for each $x \in S$ there exist at least $\alpha|A|$ elements $a \in A$ such that $a + x \in A$. Summing this over all x we see that

$$\sum_{a\in A} |\{x \in S : a + x \in A\}| \geq \alpha|A||S|.$$

Applying Cauchy–Schwarz we conclude that

$$\sum_{x,y\in S\times S} |\{a \in A : a+x, a+y \in A\}| = \sum_{a\in A} |\{x \in S : a+x \in A\}|^2 \geq \alpha^2 |A||S|^2.$$

If we set $G \subseteq S \times S$ to be all the pairs (x, y) such that

$$|\{a \in A : a+x, a+y \in A\}| \geq \alpha^2|A|/2$$

then we have

$$|A||G| \geq \sum_{(x,y)\in G} |\{a \in A : a+x, a+y \in A\}| \geq \alpha^2|A||S|^2 - \frac{\alpha^2|A|}{2}|S|^2$$

which gives (2.24). Also, if $(x, y) \in G$ then $|A \cap (A + x - y)| \geq \alpha^2|A|/2$ by definition of G, which gives (2.25). □

Before we proceed with the main theorem, we need a technical lemma that uniformizes the size of the fibers $\{(a, a') \in G : a - a' = x\}$ of $A \overset{G}{-} A$.

Lemma 2.34 (Dyadic pigeonhole principle) *Let A be an additive set, and let $G \subset A \times A$ be such that $|G| \geq \alpha|A|^2$ and $|A \overset{G}{-} A| \leq L|A|$ for some $0 < \alpha < 1$ and $L \geq 1$. Then there exists a subset G' of G with*

$$|G'| = \Omega\left(\frac{\alpha}{1 + \log\frac{1}{\alpha} + \log L}|A|^2\right)$$

and

$$|\{(a, a') \in G' : a - a' = x\}| \geq \frac{|G'|}{2|A \overset{G'}{-} A|}$$

for all $x \in A \overset{G'}{-} A$.

It is important to note that the dependence on L only enters in a logarithmic manner.

Proof Let D be the set of all x such that

$$|\{(a, a') \in \tilde{G} : a - a' = x\}| \geq \frac{\alpha|A|^2}{2L|A|} = \frac{\alpha}{2L}|A|$$

(thus D is the set of "popular differences") and set $\tilde{G}$ to be the pairs (a, a') in G such that $a - a' \in D$. Then we have $|G\backslash\tilde{G}| \leq \frac{\alpha}{2L}|A||A \overset{G}{-} A| \leq \alpha|A|^2/2$, and hence $|\tilde{G}| \geq \alpha|A|^2/2$. On the other hand, we have the crude upper bound

$$|\{(a, a') \in \tilde{G} : a - a' = x\}| \leq \sum_{a'\in A} |\{a \in A : a = x + a'\}| \leq |A|.$$

Thus if we let M be the least integer such that $2^{-M} < \frac{\alpha}{2L}$, we can partition $\tilde{G} = G_1 \cup \cdots \cup G_M$ where $G_m := \{(a, a') \in \tilde{G} : a - a' \in D_m\}$ and

$$D_m := \{x \in A \stackrel{\tilde{G}}{-} A : 2^{-m}|A| < |\{(a, a') \in \tilde{G} : a - a' = x\}| \leq 2^{-m+1}|A|\}.$$

By the pigeonhole principle, there exists $1 \leq m \leq M$ such that

$$|G_m| \geq \frac{1}{M}|G| \geq \frac{\alpha}{C\left(1 + \log\frac{1}{\alpha} + \log L\right)}|A|^2.$$

By the definition of D_m, we have

$$\frac{|G_m|}{2^{-m+1}|A|} \leq |D_m| \leq \frac{|G_m|}{2^{-m}|A|};$$

since $D_m = A \stackrel{G_m}{-} A$, we thus see that

$$|\{(a, a') \in G' : a - a' = x\}| \geq 2^{-m}|A| \geq \frac{|G'|}{2|A \stackrel{G'}{-} A|}$$

for all $x \in A \stackrel{G_m}{-} A$. The claim then follows by setting $G' := G_m$. $\square$

Now we give the main theorem of this section.

Theorem 2.35 (Asymmetric Balog–Szemerédi–Gowers theorem) *Let A, B be additive sets in an additive group Z such that $E(A, B) \geq 2\alpha|A||B|^2$ and $|A| \leq L|B|$ for some $L \geq 1$ and $0 < \alpha \leq 1$. Let $\varepsilon > 0$. Then there exists a $O_\varepsilon(\alpha^{-O_\varepsilon(1)}L^\varepsilon)$-approximate group H in Z, an additive set X in Z of cardinality $|X| = O_\varepsilon(\alpha^{-O_\varepsilon(1)}L^\varepsilon|A|/|H|)$ such that $|A \cap (X + H)| = \Omega_\varepsilon(\alpha^{O_\varepsilon(1)}L^{-\varepsilon}|A|)$, and an $x \in Z$ such that $|B \cap (x + H)| = \Omega_\varepsilon(\alpha^{O_\varepsilon(1)}L^\varepsilon|B|)$.*

Observe in the converse direction that if the conclusions of this theorem are true, then $E(A, B) = \Omega_\varepsilon(\alpha^{O_\varepsilon(1)}L^{-O(\varepsilon)}|A||B|^2)$ (Exercise 2.6.3 at the end of this section). Thus this theorem is sharp up to polynomial losses in α and L^ε, where ε can be made arbitrary small; the example in Exercise 2.6.1 can be adapted to show that this loss is necessary (Exercise 2.6.2).

Proof A direct application of Theorem 2.31 will lose far too many powers of L. The trick is to embed B in a long increasing sequence of sets $B_0, B_1, B_2, \ldots$, with each B_j being (roughly speaking) a partial difference set of the previous one, and use the pigeonhole principle to show that at some stage the ratio $|B_{j+1}|/|B_j|$ is bounded by a small power of L. One can then apply Theorem 2.31 with acceptable losses and conclude the theorem. (This method of proof is inspired by a similar argument in [40].)

We turn to the details. It will be convenient to use a variant of the Landau $O()$ and $\Omega()$ notation which can absorb factors of α and $\log L$ (which we think of as being relatively close to 1). If X, Y are non-negative quantities and j is a parameter, let us say that $X = \tilde{O}_j(Y)$ or $Y = \tilde{\Omega}_j(X)$ if one has an estimate of the form

$$X \leq C(j)\alpha^{-C(j)} Y \log^{C(j)} L$$

for some $C(j) > 0$ depending only on j.

Let $J = J(\varepsilon) \gg 1$ be a large integer to be chosen later. Let $1 > \alpha_1 > \cdots > \alpha_{J+1} > 0$ be the sequence defined recursively by $\alpha_1 := \alpha$ and $\alpha_{j+1} := \alpha_j^2/2$ for all $1 \leq j \leq J$. From induction we see that $\alpha_j = \tilde{\Omega}_j(1)$. We claim that we can find a sequence $B_0, B_1, \ldots, B_J, B_{J+1}$ of additive sets in Z with the following properties.

- $B_0 = B$, and for all $1 \leq j \leq J+1$ we have
$$B_j \subseteq \mathrm{Sym}_{\alpha_j}(A). \tag{2.26}$$
- For all $0 \leq j \leq J+1$, we have
$$\alpha_j^{-2} L|B| \geq |B_j| = \tilde{\Omega}_j(|B|). \tag{2.27}$$
- For all $0 \leq j \leq J$, there exists $G_j \subseteq B_j \times B_j$ such that
$$|G_j| = \tilde{\Omega}_j(|B_j|^2) \tag{2.28}$$
and
$$B_{j+1} = B_j \stackrel{G_j}{-} B_j. \tag{2.29}$$
Furthermore, for all $x \in B_{j+1}$ we have
$$|\{(b, b') \in G_j : b - b' = x\}| = \tilde{\Omega}_j\left(\frac{|B_j|^2}{|B_{j+1}|}\right). \tag{2.30}$$

We construct the B_j as follows. We set $B_0 := B$. From (2.22) followed by Lemma 2.34 we can construct $G_0 \subseteq B_0 \times B_0$ and $B_1 := B_0 \stackrel{G_0}{-} B_0$ obeying (2.26), (2.28), (2.29), (2.30). Since each element in $B_0 \stackrel{G_0}{-} B_0$ can be represented as a difference of a pair in G in at most $|B_0|$ ways, we have

$$|B_1| = |B_0 \stackrel{G_0}{-} B_0| \geq |G_0|/|B_0| = \tilde{\Omega}_j(|B|),$$

which is the lower bound in (2.27); the upper bound follows from (2.26) and (2.21).

Next, suppose inductively that $B_j \subseteq \mathrm{Sym}_{\alpha_j}(A)$ has already been chosen for some $1 \leq j \leq J$. Applying Lemma 2.33 (with $S := B_j$) followed by Lemma 2.34, and using the cardinality bounds already obtained in (2.27) and the construction $\alpha_{j+1}^2 := \alpha_j^2/2$ of the α_j, we can thus find $G_j \subseteq B_j \times B_j$ and $B_{j+1} := B_j \stackrel{G_j}{-} B_j$

obeying (2.26), (2.28), (2.29), (2.30). This closes the induction and so we can construct the B_j for all $0 \leq j \leq J+1$, and similarly obtain the G_j for all $1 \leq j \leq J$.

Now for the crucial step (which explains why we iterated the above procedure so many times). From (2.27) and the pigeonhole principle, there exists $1 \leq j \leq J$ such that

$$|B_{j+1}| = \tilde{O}_J\big(L^{O(1/J)}|B_j|\big);$$

the point is that we have managed to replace L by the substantially smaller quantity $L^{O(1/J)}$. If we now apply (2.29), (2.28), and Theorem 2.31, we can thus find a $\tilde{O}_J(L^{O(1/J)})$-approximate group H of cardinality

$$|H| = \tilde{O}_J\big(L^{O(1/J)}|B_j|\big) \tag{2.31}$$

and an $x_j \in Z$ such that

$$|B_j \cap (H + x_j)| = \tilde{\Omega}_J\big(L^{-C_0/J}|B_j|\big) \tag{2.32}$$

for some absolute constant C_0. It remains to relate H to B and to A. We begin with B. From (2.32) and (2.30) (with j replaced by $j-1$) we have

$$|\{(b, b') \in G_{j-1} : b - b' \in B_j \cap (H + x_j)\}| = \tilde{\Omega}_J\big(L^{-C_0/J}|B_{j-1}|^2\big),$$

so in particular

$$|\{(b, b') \in B_{j-1} \times B_{j-1} : b \in H + x_j + b'\}| = \tilde{\Omega}_J\big(L^{-C_0/J}|B_{j-1}|^2\big).$$

Thus by the pigeonhole principle, there exists a b' such that

$$|\{b \in B_{j-1} : b - b' \in H + x_j + b'\}| = \tilde{\Omega}_J\big(L^{-C_0/J}|B_{j-1}|\big).$$

Thus if we set $x_{j-1} := x_j + b'$ then we have

$$|B_{j-1} \cap (H + x_{j-1})| = \tilde{\Omega}_J\big(L^{-C_0/J}|B_{j-1}|\big). \tag{2.33}$$

We now repeat this argument with j replaced by $j-1$ and (2.32) replaced by (2.33). Iterating this at most J times, we eventually locate an $x = x_0 \in Z$ such that

$$|B \cap (H + x)| = \tilde{\Omega}_J\big(L^{-C_0/J}|B|\big),$$

which gives the desired control on B if J is sufficiently large depending on ε.

It remains to control A. From (2.32), (2.31) and (2.26) we have

$$|\{y \in H + x_j : y \in \mathrm{Sym}_{\alpha_j}(A)\}| = \tilde{\Omega}_J\big(L^{-O(1/J)}|H|\big)$$

and thus by definition of $\mathrm{Sym}_{\alpha_j}(A)$ and α_j

$$|\{(a, y) \in A \times (H + x_j) : a + y \in A\}| = \tilde{\Omega}_J\big(L^{-O(1/J)}|H||A|\big).$$

We rewrite this as

$$\sum_{x \in x_j + A} |A \cap (H + x)| = \tilde{\Omega}_J\big(L^{-O(1/J)}|H||A|\big).$$

We can therefore find a subset X_0 of $x_j + A$ with

$$|X_0| = \tilde{\Omega}_J\big(L^{-O(1/J)}|A|\big) \tag{2.34}$$

such that

$$|A \cap (H + x)| = \tilde{\Omega}_J\big(L^{-O(1/J)}|H|\big) \text{ for all } x \in X_0.$$

Now we use an argument similar to that used to prove Ruzsa's covering lemma (Lemma 2.14). Let X be a subset of X_0 such that the sets $\{H + x : x \in X\}$ are all disjoint, and which is maximal with respect to set inclusion. Then we have

$$|A \cap (H + X)| = \sum_{x \in X} |A \cap (H + x)| = \tilde{\Omega}_J\big(L^{-O(1/J)}|H||X|\big). \tag{2.35}$$

On the other hand, if $y \in X_0$, then by maximality of X there exists $x \in X$ such that $x + H$ intersects $y + H$. In other words, X_0 is covered by $X + H - H$, and hence (since H is a $\tilde{O}(L^{O(1/J)})$-approximate group)

$$|X_0| \leq |X||H - H| = \tilde{O}\big(|X|L^{O(1/J)}|H|\big). \tag{2.36}$$

Combining (2.34), (2.35), (2.36) we see that X obeys all the desired properties, if J is chosen sufficiently small depending on ε. □

The above theorem can also be put in a form resembling Theorem 2.29:

Corollary 2.36 *Let A, B be additive sets with common ambient group such that $E(A, B) \geq 2\alpha|A||B|^2$ and $|A| \leq L|B|$ for some $L \geq 1$ and $0 < \alpha \leq 1$. Let $\varepsilon > 0$. Then there exists subsets $A' \subseteq A$ and $B' \subseteq B$ such that*

$$|A'| = \Omega_\varepsilon\big(\alpha^{O_\varepsilon(1)}L^{-\varepsilon}|A|\big)$$
$$|B'| = \Omega_\varepsilon\big(\alpha^{O_\varepsilon(1)}L^{-\varepsilon}|B|\big)$$
$$|A' + nB' - mB'| = O_\varepsilon\big(\alpha^{-O_\varepsilon(1)}L^{\varepsilon}\big)^{n+m}|A|$$

for all integers $n, m \geq 0$.

Proof Apply Theorem 2.35 and set $A' := A \cap (X + H)$ and $B' := B \cap (x + H)$. □

Because of (2.8), the above results give some partial results concerning the situation when $|A + B| \leq K|A|$ and $|A|$ is much larger than $|B|$, but these results will be rather weak. We will give a better result concerning this problem in Section 6.5, once we develop the Plünnecke inequalities.

Exercises

2.6.1 [297] Let n be a large integer, and let $Z := \mathbf{Z}^{2n}$. Let A be the additive set

$$A := \{(x_1, x_2, \dots, x_{2n}) \in \mathbf{Z}^{2n} : x_1 + \cdots + x_{2n} = n; x_1, \dots, x_{2n} \geq 0\}$$

and let $B := \{e_1, \dots, e_{2n}\}$. Show that $|B| = 2n$, that $|A| = (27/4)^{n+o(1)}$, that $|A + B| = O(|A|)$, but that $|A - B| \geq n|A|$. (You may find Stirling's formula (1.52) to be useful.)

2.6.2 Modify Exercise 2.6.1 to show that one cannot take $\varepsilon = 0$ in Theorem 2.35.

2.6.3 Let A, B be additive sets and let $\varepsilon > 0, 0 < \alpha < 1$, and $L \geq 1$ be such that the conclusions of Theorem 2.35 are satisfied. Conclude that $E(A, B) = \Omega_\varepsilon(\alpha^{O_\varepsilon(1)} L^{-O(\varepsilon)} |A||B|^2)$.

2.6.4 Let A be an additive set. By modifying the proof of Lemma 2.13, establish the inequality

$$|A - A + n\mathrm{Sym}_\alpha(A)| \leq \frac{\delta[A]^{n+1}}{\alpha^n}|A|$$

for all integers $n \geq 0$ and all $0 < \alpha < 1$.

2.6.5 [220] Let A be an additive set such that $A - A$ is not a group. Show that there exists $h \in A - A$ such that $1 \leq |A \cap (A + h)| \leq |A|/2$. (Hint: argue by contradiction, and analyze $\mathrm{Sym}_\alpha(A)$ for some α slightly greater than 1/2.) Conclude in particular that if $|A - A| < \frac{3}{2}|A|$, then $A - A$ is a group. Note that the example $A = \{0, 1\} \subset \mathbf{Z}$ shows that the constant $\frac{3}{2}$ cannot be improved; one can also make this example larger, for instance by taking the Cartesian product of $\{0, 1\}$ with a finite group. For a more refined estimate on $A - A$, see Theorem 5.5 and Corollary 5.6.

2.6.6 Let A, B be additive sets with common ambient group such that $|A + B| \leq K|A|$ and $|A| \leq L|B|$ for some $K, L \geq 1$. Let $\varepsilon > 0$. Show that there exists a $O_\varepsilon(K^{O_\varepsilon(1)} L^\varepsilon)$-approximate group H such that B is contained in a translate of H, and that A is contained in at most $O_\varepsilon(K^{O_\varepsilon(1)} L^\varepsilon |A|/|H|)$ translates of H; compare this with Proposition 2.2. (Hint: Apply Theorem 2.35 and the Ruzsa covering lemma (Lemma 2.14).)

2.6.7 Let A be an additive set, and let B be a subset of A such that $|B| \geq (1 - \varepsilon)|A|$ for some $0 < \varepsilon < 1$. Prove that

$$\mathrm{Sym}_{\alpha/(1-\varepsilon)}(B) \subseteq \mathrm{Sym}_\alpha(A) \subseteq \mathrm{Sym}_{(\alpha-2\varepsilon)/(1-\varepsilon)}(B)$$

for every $\alpha \in \mathbf{R}$.