3.1 Additive groups

We first review the theory of additive groups, which we introduced in Definition 0.1, obtaining in particular the classification theorem for finitely generated additive groups (Corollary 3.9). This is a fundamental result in additive group theory, but it will also motivate similar results concerning other additively structured sets such as progressions, Bohr sets, and the intersection of convex sets and lattices.

Typical examples of additive groups include the integers $\mathbf{Z}$, the reals $\mathbf{R}$, the lattices $\mathbf{Z}^d$, the Euclidean spaces $\mathbf{R}^d$, the torus groups $\mathbf{R}^d/\mathbf{Z}^d$, and the cyclic groups $\mathbf{Z}_N := \mathbf{Z}/N \cdot \mathbf{Z}$. Note that the direct sum $Z \oplus Z'$ of two additive groups is again an additive group. We now make an important distinction between torsion groups and torsion-free groups.

Definition 3.1 (Torsion) If Z is an additive group and $x \in Z$, we let $\mathrm{ord}(x)$ be the least integer $n \geq 1$ such that $n \cdot x = 0$, or $\mathrm{ord}(x) = +\infty$ if no such integer exists. We say that Z is a *torsion group* if $\mathrm{ord}(x)$ is finite for all $x \in Z$, and we say that it is an *r-torsion group* for some $r \geq 1$ if $\mathrm{ord}(x)$ divides r for all $x \in Z$. We say that Z is *torsion-free* if $\mathrm{ord}(x) = +\infty$ for all $x \in Z$.

Examples 3.2 The groups $\mathbf{Z}, \mathbf{R}, \mathbf{Z}^d, \mathbf{R}^d$ are torsion-free, whereas any finite group such as $\mathbf{Z}_N$ is a torsion group.

A *group homomorphism* $\phi : Z \to Z'$ between two additive groups Z, Z' is any map which preserves addition, negation, and zero (thus $\phi(x + y) = \phi(x) + \phi(y)$, $\phi(-x) = -\phi(x)$, and $\phi(0) = 0$ for all $x, y \in Z$). If ϕ is also invertible, then the inverse ϕ^{-1} is automatically a group homomorphism, and we say that ϕ is an *group isomorphism*, and Z and Z' are *group isomorphic*. Since all of our notions here shall be defined in terms of the addition, negation, and zero operations, they will all be preserved by group isomorphism, and so we will treat group isomorphic groups to be essentially equivalent. Later on we shall develop a weaker notion of *Freiman homomorphism* and *Freiman isomorphism* which is more suitable for the study of "approximate groups" (sets that are "almost" closed under addition); see Section 5.3.

If G is a subgroup of an additive group Z, then we can form the *quotient group*

$$Z/G := \{x + G : x \in Z\}$$

formed by taking all the cosets of G; this is easily verified to be a group (though it is no longer a subgroup of Z). For instance, the cyclic group $\mathbf{Z}_N = \mathbf{Z}/(N \cdot \mathbf{Z})$ is the quotient of the integers $\mathbf{Z}$ by the subgroup $N \cdot \mathbf{Z}$. Observe that the map $\pi : Z \to Z/G$ defined by $\pi(x) := x + G$ is a surjective homomorphism.