3.2.6 Let P be a proper progression of rank d, and let A be a subset of P such that $|A| \leq \varepsilon |P|$ for some $0 < \varepsilon < 1$. Show that $P \backslash A$ contains a proper progression Q of rank d with $|Q| \geq C^{-d}/\varepsilon$ for some absolute constant C.

3.2.7 Let A be an additive set in an ambient group Z, and let $v \in Z$. Show that $|(A+v)\backslash A| \leq 1$ if and only if A is equal to a proper arithmetic progression of step v, union a finite (possibly zero) number of translates of the group $\langle v \rangle$. In particular, if $|A| < \mathrm{ord}(v)$, then $|(A+v)\backslash A| > 0$, and $|(A+v)\backslash A| = 1$ if and only if A is a proper arithmetic progression of step v.

3.3 Convex bodies

We now review some of the theory of convex bodies in $\mathbf{R}^d$, which are in some sense the continuous analogue of generalized arithmetic progressions. This is of course a vast field, and we shall restrict ourselves with just a small sample of results, relating to the additive theory of such sets, to covering lemmas, and the relationship between addition and volume.

We shall use $\mathrm{mes}(A)$ to denote the volume of a set A in $\mathbf{R}^d$; to avoid issues with measurability we shall mostly concern ourselves with bounded open sets A. If $A \in \mathbf{R}^d$ and $\lambda \in \mathbf{R}$, we use $\lambda \cdot A$ to denote the dilation $\lambda \cdot A := \{\lambda x : x \in A\}$. Observe that $\mathrm{mes}(\lambda A) = |\lambda|^d \mathrm{mes}(A)$.

Recall that a set A in $\mathbf{R}^d$ is *convex* if we have $(1-\theta)x + \theta y \in A$ whenever $x, y \in A$ and $0 \leq \theta \leq 1$; equivalently, a set is convex if and only if

$$a \cdot A + b \cdot A = (a+b) \cdot A$$

for all real $a, b \geq 0$ (Exercise 3.3.3). In particular we have $nA = |n| \cdot A$ for any integer A. We call A a *convex body* if it is convex, open, non-empty, and bounded. In particular we see that if A is a convex body, then

$$\mathrm{mes}(A+A) = \mathrm{mes}(2 \cdot A) = 2^d \mathrm{mes}(A), \tag{3.4}$$

so convex bodies have small doubling constant. As for $A - A$, we can use

Lemma 3.12 *[297] For any bounded open subsets A, B, C of $\mathbf{R}^d$ (not necessarily convex), we have*

$$\mathrm{mes}(A-C)\mathrm{mes}(B) \leq \mathrm{mes}(A-B)\mathrm{mes}(B-C).$$

This is proven by modifying the proof of Lemma 2.6 appropriately and is left as an exercise. From this Lemma (with $A = C$ and $B = -A$) and (3.4) we obtain

$$\mathrm{mes}(A - A) \leq 4^d \mathrm{mes}(A); \tag{3.5}$$

compare these bounds with Lemma 3.10. For a slight refinement of (3.5), see Exercise 3.4.6. In the converse direction, the Brunn–Minkowski inequality (Theorem 3.16 below) will give $\mathrm{mes}(A - A) \geq 2^d \mathrm{mes}(A)$.

Call a convex body A *symmetric* if $A = -A$; thus for us symmetry will always be with respect to the origin. The following theorem of John essentially classifies all convex bodies (symmetric and non-symmetric) up to a (dimension-dependent) constant factor.

Theorem 3.13 (John's theorem) *[194] Let A be a convex body in $\mathbf{R}^d$. Then there exists an invertible linear transformation $T : \mathbf{R}^d \to \mathbf{R}^d$ on $\mathbf{R}^d$ and a point $x_0 \in A$ such that*

$$B_d \subseteq T(A - x_0) \subseteq d \cdot B_d,$$

where B_d is the unit ball $\{(x_1, \ldots, x_d) \in \mathbf{R}^d : x_1^2 + \cdots + x_d^2 < 1\}$. If A is symmetric, then we can improve these inclusions to

$$B_d \subseteq T(A) \subseteq \sqrt{d} \cdot B_d.$$

The constants d and $\sqrt{d}$ are sharp; see the exercises.

Proof We will use a variational argument. Define an *ellipsoid* to be any set E of the form $E = L(B_d) + x_0$, where B_d is the unit ball, $x_0 \in \mathbf{R}^d$, and L is a (possibly degenerate) linear transformation in $\mathbf{R}^d$; we allow the ellipsoid to be degenerate for compactness reasons. Since A is open and bounded, it is easy to see that the set of all ellipsoids E contained in A is a compact set (with respect to the usual topology on L and x_0). Also the volume of the ellipsoid E is $\mathrm{mes}(E) = |\det(L)|$, whch is clearly a continuous function of E. Thus there exists an ellipsoid $E = L(B_d) + x_0$ in A which maximizes the volume $\mathrm{mes}(E)$; since A is open, this volume is non-zero, and hence L is invertible. By applying L^{-1} if necessary (observing that the conclusion of the lemma is invariant under invertible linear transformations) we may thus assume that E is a translate $E = B_d + y_0$ of the unit ball, where $y_0 = L^{-1}(x_0)$.

Let us now restrict to the case where A is symmetric. Observe that if A contains $B_d + y_0$ then it also contains $B_d - y_0$ by symmetry, and hence contains B_d, which is in the convex hull of $B_d + y_0$ and $B_d - y_0$. To conclude the proof of the lemma in this case we need to show that A is contained in $\sqrt{d} \cdot B_d$. Suppose for contradiction that A was not contained in $\sqrt{d} \cdot B_d$; without loss of generality (and using the hypothesis that A is open) we may then suppose that $re_1 \in A$ for some $r > \sqrt{d}$, where e_1 is the first basis vector. Observe now from elementary geometry that if ω is any point on the boundary of B_d making an angle $\angle(\omega, e_1) < \arctan(r^2 - 1)$, then the line segment connecting ω to re_1 is disjoint from (and not tangent to) B_d, and, since B_d and re_1 both lie in the convex set A, we thus see that ω also lies in the open set A. By symmetry, the same is true if $\angle(\omega, -e_1) < \arctan(\sqrt{r^2 - 1})$.

We now perturb the ball B_d by an epsilon. Now let $\delta > 0$ be a small number, let $\varepsilon > 0$ be an even smaller one, and consider the ellipsoid $L_{\varepsilon,\delta}(B_d)$, where

$$L_{\varepsilon,\delta}(x_1, \ldots, x_d) := ((1 + (\sqrt{d} - 1 + \delta)\varepsilon)x_1, (1 - \varepsilon)x_2, \ldots, (1 - \varepsilon)x_d)).$$

When $\varepsilon = 0$, $L_{\varepsilon,\delta}(B_d)$ is just B_d. Now consider how $L_{\varepsilon,\delta}(B_d)$ evolves in ε. The determinant of this transformation is $(1 + (\sqrt{d} - 1 + \delta)\varepsilon)(1 - \varepsilon)^{d-1}$, which has a positive ε-derivative at $\varepsilon = 0$. Thus $L_{\varepsilon,\delta}(B_d)$ has larger volume than B for sufficiently small ε (depending on δ). Now we check which points on the surface of $L_{\varepsilon,\delta}(B_d)$ expand away from the origin, and which ones contract. A simple computation shows that for any $\omega = (\omega_1, \ldots, \omega_d)$ on the boundary of B_d, the derivative

$$\frac{d}{d\varepsilon} \|L_{\varepsilon,\delta}(\omega)\|^2 \Big|_{\varepsilon=0},$$

where $\|(y_1, \ldots, y_d)\|^2 := y_1^2 + \cdots + y_d^2$, is negative unless

$$(d - 1 + \delta)\omega_1^2 - \omega_2^2 - \cdots - \omega_d^2 \geq 0,$$

or in other words that

$$\angle(\omega, \pm e_1) \leq \arctan(\sqrt{d - 1 + \delta}).$$

But if δ is small enough depending on r, this region is contained entirely within the interior of A by the previous discussion. Thus for ε small enough $L_{\varepsilon,\delta}(B_d)$ is completely contained inside A but has larger volume, contradicting the maximality of B_d, and we are done.

Now suppose that A is not symmetric. In this case we may translate so that $y_0 = 0$. Thus again we have $B_d \subseteq A$, and the task is to show that $A \subseteq n \cdot B_d$. Suppose again for contradiction that $re_1 \in A$ for some $r > d$; again this means that every point ω in the boundary of B_d with $\angle(\omega, e_1) < \arctan(r^2 - 1)$ will lie in the interior of A. Now let $\delta, \varepsilon > 0$ and consider the ellipsoid

$$L_{\varepsilon,\delta}(x_1, \ldots, x_d) + (d - 1 + \delta)\varepsilon e_1;$$

again, this ellipsoid has larger volume than B_d if ε is sufficiently small. Also, we see that

$$\frac{d}{d\varepsilon} \|L_{\varepsilon,\delta}(\omega) + (d - 1 + \delta)\varepsilon e_1\|^2 \Big|_{\varepsilon=0}$$

is negative unless

$$(d-1+\delta)\omega_1^2+(d-1+\delta)\omega_1-\omega_2^2-\cdots-\omega_d^2 \geq 0,$$

which can be rewritten (using $\|\omega\|=1$) as

$$((d+\delta)\omega_1-1)(\omega_1+1)\geq 0,$$

or equivalently

$$\angle(\omega, e_1) \leq \arctan(\sqrt{(d+\delta)^2-1}).$$

We now argue as in the symmetric case to obtain again the desired contradiction, if δ is chosen so that $d+\delta < r$. $\square$

As a corollary of Theorem 3.13 we see that if A is a convex body, we can cover $A+A$ or $A-A$ by a relatively small number of copies of A:

$$A \pm A \text{ can be covered by } O(d)^d \text{ translates of } A. \tag{3.6}$$

This follows immediately from the geometric observation that $d\cdot B_d + d\cdot B_d = 2d\cdot B_d$ can be covered by $O(d)^d$ translates of B_d. If A is symmetric, we can improve this somewhat. In the special case when A is a cube or a box, it is clear that $A\pm A$ can be covered by 2^d translates of A (cf. Lemma 3.10), but one cannot hope for this in general; for instance if A is a disk in $\mathbf{R}^2$ then one needs six copies of A to cover $A\pm A$. In the general case, we will need the following continuous version of

Lemma 3.14 (Ruzsa's covering lemma) *[300], [250] For any bounded subsets A, B of $\mathbf{R}^d$ with positive measure (not necessarily convex), we can cover B by at most* $\min(\frac{\mathrm{mes}(A+B)}{\mathrm{mes}(A)}, \frac{\mathrm{mes}(A-B)}{\mathrm{mes}(A)})$ *translates of $A-A$.*

The proof of this lemma is nearly identical to that of Lemma 2.14 and is left as an exercise. As a consequence we can improve (3.6) for symmetric convex bodies:

Corollary 3.15 *Let $A\subset\mathbf{R}^d$ be a convex body, and let $\lambda,\mu>0$ be real. Then $\lambda\cdot A$ can be covered by at most $(\lambda+1)^d$ translates of $A-A$, and $\lambda\cdot A-\mu\cdot A$ can be covered by $(2\max(\lambda,\mu)+1)^d$ translates of $A-A$. If A is symmetric, then $\lambda\cdot A$ can be also covered by $(2\lambda+1)^d$ translates of A.*

Proof The first claim follows from Lemma 3.14 since $\mathrm{mes}(\lambda\cdot A+A)=(\lambda+1)^d\mathrm{mes}(A)$. To prove the second claim, we may take $\lambda\geq\mu$. The first claim implies that $2\lambda\cdot A$ can be covered by $(2\lambda+1)^d$ translates of $A-A=2\cdot A$, and the third claim follows by rescaling by $1/2$. Finally, the second claim follows by applying the third claim to $A-A$. $\square$

Observe that all the bounds obtained here tend to be exponential in d or worse. Thus when using the theory of convex bodies to obtain explicit estimates, it is often important to keep the dimension d as low as possible, even at the cost of making some other parameters larger than would otherwise be necessary. See [250] for further discussion of sum set and covering estimates for convex bodies.

We have not yet seen what happens to the sum or difference of two unrelated convex bodies A and B. The relationship here is given by the *Brunn–Minkowski inequality*, which we turn to next.

Exercises

3.3.1 Prove Lemma 3.12.

3.3.2 Prove Lemma 3.14.

3.3.3 Verify that the two definitions of convexity given are indeed equivalent.

3.3.4 Let A be an open bounded subset of $\mathbf{R}^d$. Show that A is convex if and only if $2A = 2 \cdot A$, and that A is convex and symmetric if and only if $2A = -2 \cdot A$.

3.3.5 For any $s > 0$ let $\Gamma(s) := \int_0^\infty e^{-x} x^{s-1}\, dx$ denote the Gamma function. Show that $\Gamma(s+1) = s\Gamma(s)$ for all $s > 0$, that $\Gamma(d) = (d-1)!$ for all $d \geq 1$, that $\Gamma(1/2) = \sqrt{\pi}$, and we have the Stirling formula

$$\log \Gamma(s) = s \log s - s + O(\log s) \tag{3.7}$$

for all large s. (Hint: use (1.52) and the monotonicity of the Γ function.)

3.3.6 Let B_d be the unit ball in $\mathbf{R}^d$. By evaluating the integral $\int_{\mathbf{R}^d} e^{-\pi|x|^2}\, dx$ in both Cartesian and polar coordinates, and using the preceding exercise, establish the volume formula

$$\operatorname{mes}(B_d) = \frac{\Gamma(3/2)^d 2^d}{\Gamma(d/2+1)} = (2\pi e + o(1))^{d/2} d^{-d/2}. \tag{3.8}$$

3.3.7 Let O_d be the octahedron given by the convex hull of $\pm e_1, \ldots, \pm e_d$ in $\mathbf{R}^d$. Show that $\operatorname{mes}(O_d) = 2^d/d! = (2e + o(1))^d d^{-d}$. Thus in large dimension the octahedron becomes considerably smaller than the circumscribing ball B_d which contains it, which in turn is considerably smaller than the circumscribing cube.

3.3.8 Show that the constants d and $\sqrt{d}$ in Theorem 3.13 cannot be improved. (For the non-symmetric case, take A to be a d-simplex (the convex hull of d points in $\mathbf{R}^d$); for the symmetric case, take A to be a cube.)

3.3.9 If A and A' are two symmetric convex bodies in $\mathbf{R}^d$, show that there exists an invertible linear transformation $T : \mathbf{R}^d \to \mathbf{R}^d$ such that

$$A \subseteq T(A') \subset d \cdot A.$$