State and prove a similar result in the case when A and A' are not necessarily symmetric.

3.3.10 Let A, B be open bounded sets. Show that

$$\text{mes}((A - A) \cap (B - B)) \geq \frac{\text{mes}(A)\text{mes}(B)}{\text{mes}(A \pm B)}$$

for either choice of sign $\pm$, by developing a continuous analogue of the arguments used to prove (2.8). (Alternatively, one can try to discretize A and B to replace them with finite sets, and then use (2.8) directly.)

3.3.11 [26] Let A be a symmetric convex body in $\mathbf{R}^d$, which contains the ball $\rho \cdot B$ of radius $\rho > 0$ centered at the origin. Let V be any r-dimensional subspace of $\mathbf{R}^d$. Show that $\text{mes}_r(A \cap V) \leq \frac{d!}{r!(2\rho)^{d-r}}\text{mes}_d(A)$, where mes_r denotes r-dimensional measure. (Hint: first show that if $r < d$, then there exists an $r + 1$-dimensional space V_1 containing V such that mes_{r+1} $(A \cap V_1) \geq \frac{2\rho}{r+1}\text{mes}_r(A \cap V)$. Then continue inductively.)

3.4 The Brunn–Minkowski inequality

The purpose of this section is to prove the following lower bound for the volume $\text{mes}(A + B)$ of a sum set.

Theorem 3.16 (Brunn–Minkowski inequality) *If A and B are non-empty bounded open subsets of $\mathbf{R}^d$, then*

$$\text{mes}(A + B)^{1/d} \geq \text{mes}(A)^{1/d} + \text{mes}(B)^{1/d}.$$

This inequality is sharp (Exercise 3.4.2). The theorem also applies if A and B are merely measurable (as opposed to being bounded and open), though one must then also assume that $A + B$ is measurable; we will not prove this here. In general, there is no upper bound for $\text{mes}(A + B)$; consider for instance the case when A is the x-axis and B is the y-axis in $\mathbf{R}^2$, then A, B both have measure zero but $A + B$ is all of $\mathbf{R}^2$. One can easily modify this example to show that there is no upper bound for $\text{mes}(A + B)$ in terms of $\text{mes}(A)$ and $\text{mes}(B)$ when A, B are bounded open sets. See [128] for a thorough survey of the Brunn–Minkowski inequality and related topics.

To prove this theorem, it suffices to prove the following dimension-independent version:

Theorem 3.17 *If A and B are non-empty bounded open subsets of $\mathbf{R}^d$, and $0 < \theta < 1$, then*

$$\text{mes}((1 - \theta) \cdot A + \theta \cdot B) \geq \text{mes}(A)^{1-\theta}\text{mes}(B)^{\theta}.$$

To see why Theorem 3.17 implies the Brunn–Minkowski inequality, apply Theorem 3.17 with A and B replaced by $\mathrm{mes}(A)^{-1/d} \cdot A$ and $\mathrm{mes}(B)^{-1/d} \cdot B$ to obtain

$$\mathrm{mes}\left(\frac{1-\theta}{\mathrm{mes}(A)^{1/d}} \cdot A + \frac{\theta}{\mathrm{mes}(B)^{1/d}} \cdot B\right) \geq 1$$

for any $0 < \theta < 1$. Setting

$$\theta := \frac{\mathrm{mes}(B)^{1/d}}{\mathrm{mes}(A)^{1/d} + \mathrm{mes}(B)^{1/d}}$$

we obtain the result. Conversely, one can easily deduce Theorem 3.17 from the Brunn–Minkowski inequality (Exercise 3.4.1).

It remains to prove Theorem 3.17. We begin by first proving

Lemma 3.18 (One-dimensional Brunn–Minkowski inequality) *If A and B are non-empty bounded open subsets of $\mathbf{R}$, then* $\mathrm{mes}(A + B) \geq \mathrm{mes}(A) + \mathrm{mes}(B)$.

Proof The hypotheses and conclusion of this lemma are invariant under independent translations of A and B, so we can assume that $\sup(A) = 0$ and $\inf(B) = 0$, hence in particular A and B are disjoint. But then we see that $A + B$ contains both A and B separately, and we are done. □

Using this Lemma, we deduce

Proposition 3.19 (One-dimensional Prékopa–Leindler inequality) *Let $0 < \theta < 1$, and let $f, g, h : \mathbf{R} \to [0, \infty)$ be lower semi-continuous, compactly supported non-negative functions on $\mathbf{R}$ such that*

$$h((1-\theta)x + \theta y) \geq f(x)^{1-\theta} g(y)^{\theta}$$

for all $x, y \in \mathbf{R}$. Then we have

$$\int_{\mathbf{R}} h \geq \left(\int_{\mathbf{R}} f\right)^{1-\theta} \left(\int_{\mathbf{R}} g\right)^{\theta}.$$

Proof By multiplying f, g, h by appropriate positive constants we may normalize $\sup_x f(x) = \sup_y f(y) = 1$.

Let $1 > \lambda > 0$ be arbitrary. Observe that if $f(x) > \lambda$ and $g(y) > \lambda$, then by hypothesis $h((1-\theta)x + \theta y) > \lambda$. Thus we have

$$\{z \in \mathbf{R} : h(z) > \lambda\} \subseteq (1-\theta) \cdot \{x \in \mathbf{R} : f(x) > \lambda\} + \theta \cdot \{y \in \mathbf{R} : g(y) > \lambda\}.$$

Since f, g, h are lower semi-continuous and compactly supported, all the sets above are open and bounded, hence by Lemma 3.18

$$\mathrm{mes}(\{z \in \mathbf{R} : h(z) > \lambda\}) \geq (1-\theta)\mathrm{mes}(\{x \in \mathbf{R} : f(x) > \lambda\}) + \theta\mathrm{mes}(\{y \in \mathbf{R} : g(y) > \lambda\}).$$

Integrating this for $\lambda \in [0, \infty)$ and using Fubini's theorem (cf. (1.6)), the claim follows from the arithmetic mean–geometric mean inequality. □

Now we iterate this to higher dimensions.

Proposition 3.20 (Higher-dimensional Prékopa–Leindler inequality) *Let $0 < \theta < 1$, $d \geq 1$, and let $f, g, h : \mathbf{R}^d \to [0, \infty)$ be lower semi-continuous, compactly supported non-negative functions on $\mathbf{R}^d$ such that*

$$h((1-\theta)x + \theta y) \geq f(x)^{1-\theta} g(y)^{\theta}$$

for all $x, y \in \mathbf{R}^d$. Then we have

$$\int_{\mathbf{R}} h \geq \left(\int_{\mathbf{R}} f\right)^{1-\theta} \left(\int_{\mathbf{R}} g\right)^{\theta}.$$

Proof We induce on d. When $d = 1$ this is just Proposition 3.19. Now assume inductively that $d > 1$ and the claim has already been proven for all smaller dimensions d. Define the one-dimensional function $h_d : \mathbf{R} \to [0, \infty)$ by

$$h_d(x_d) := \int_{\mathbf{R}^{d-1}} h(x_1, \ldots, x_d)\, dx_1 \cdots dx_{d-1},$$

and similarly define f_d, g_d. One can easily check that (using Fatou's lemma) that these functions are lower semi-continuous and compactly supported. Also, applying the inductive hypothesis at dimension $d - 1$ we see that

$$h_d((1-\theta)x_d + \theta y_d) \geq f_d(y_d)^{1-\theta} g_d(y_d)^{\theta}$$

for all $x_d, y_d \in \mathbf{R}$. If we then apply the one-dimensional Prékopa–Leindler inequality, we obtain the desired result. □

If we apply Proposition 3.20 with $f := 1_A$, $g := 1_B$, and $h := 1_{(1-\theta)A+\theta B}$ we obtain Theorem 3.17, and the Brunn–Minkowski inequality follows.

Exercises

3.4.1 Show that Theorem 3.16 implies Theorem 3.17.

3.4.2 Show that equality in Theorem 3.17 can occur when A is convex, and $B = \lambda \cdot A + x_0$ for some $\lambda, x_0 \in \mathbf{R}^n$. Conversely, if A and B are non-empty bounded open subsets of $\mathbf{R}^d$, show that the preceding situation is in fact the only case in which equality can be attained. (The case when A and B are merely measurable is a bit trickier, and is of course only true up to sets of measure zero; see [128] for further discussion).

3.4.3 Let A be a convex body in $\mathbf{R}^d$. Using Theorem 3.17, show that the cross-sectional areas $f(x_d) := \mathrm{mes}(\{x' \in \mathbf{R}^{d-1} : (x', x_d) \in A\})$ are a