## 4.5 $\Lambda(p)$ constants, $B_h[g]$ sets, and dissociated sets

In Section 4.3 we discussed one Fourier-analytic characteristic of an additive set $A$ in a finite additive group $Z$, namely its linear bias. In this section we discuss a rather different characteristic, namely the $\Lambda(p)$ constants of a set $S$ of frequencies. These constants measure how "dissociated" or "Sidon-like" a set[1] $S$ is; in more practical terms, the $\Lambda(p)$ constants quantify the independence of the characters associated to $S$ in a certain $L^p(Z)$ sense. These constants can be used to obtain precise control on the arithmetic structure of $S$, for instance in controlling iterated sum sets of $S$. One feature of these constants is that they are stable under passage to subsets, thus $\Lambda(p)$ constants will also control iterated sum sets of subsets $S'$ of $S$. This stability (which is not present in the Fourier bias, unless one takes random subsets as in Lemma 4.16) is useful for a number of applications.

We begin with the formal definition of the $\Lambda(p)$ constants.

**Definition 4.26** **($\Lambda(p)$ constants)** Let $S$ be an additive set in a finite[2] additive group $Z$, and let $2 \leq p \leq \infty$. We define the $\Lambda(p)$ *constant* of $S$, denoted $\|S\|_{\Lambda(p)}$, to be the best constant such that the inequality

$$\left\| \sum_{\xi \in S} c(\xi) e(\xi \cdot x) \right\|_{L^p(Z)} \leq \|S\|_{\Lambda(p)} \|c\|_{l^2(S)} \tag{4.30}$$

holds for all sequences $c : S \to \mathbf{C}$ of complex numbers.

One can easily establish the bound

$$\|S\|_{\Lambda(p)} \leq |S|^{1/2-1/p}, \tag{4.31}$$

for $2 \leq p \leq \infty$, with equality at the endpoints $p = 2, \infty$; see Exercise 4.5.6. This exercise indicates that largeness of $\Lambda(p)$ constants is correlated to strong additive structure of $S$. At the other extreme, we now show that smallness of $\Lambda(p)$ constants is correlated to strong lack of additive structure of $S$.

**Definition 4.27** **($B_h$ sets)** Let $h \geq 2$. A non-empty subset $S$ of an additive group $Z$ is a $B_h$ *set* if for any $\xi_1, \ldots, \xi_h, \eta_1, \ldots, \eta_h \in S$, one has $\xi_1 + \cdots + \xi_h = \eta_1 + \cdots + \eta_h$ if and only if $(\xi_1, \ldots, \xi_h)$ is a permutation of $(\eta_1, \ldots, \eta_h)$. We say $S$ is a *Sidon set* if it is a $B_2$ set.

These sets are the $g = 1$ version of the $B_h[g]$ sets, encountered in Section 1.7.1; Sidon sets were also briefly mentioned in Section 2.2. Note that we do not bother with the notion of a $B_1$ set, since every set is trivially a $B_1$ set.

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[1] Here, we use "Sidon set" to denote a set whose pairwise sums are all disjoint. There is another, more Fourier-analytic, notion of a Sidon set related to $\Lambda(p)$ constants which we will not discuss here.

[2] One can also define the concept of a $\Lambda(p)$ constant for subsets of the integers, or more general additive groups, but we will not need to do so in this book.

**Example 4.28** For any $M > 1$, the set $S := \{0\} \cup (M^\wedge\mathbf{N}) = \{0, 1, M, M^2, \ldots\}$ is a $B_h$ set in $\mathbf{Z}$ if and only if $h < M$. In particular, the powers of 2 form a Sidon set. One can of course truncate these examples to finite additive groups such as $\mathbf{Z}_N$; note that any non-empty subset of a $B_h$ set is also a $B_h$ set.

**Proposition 4.29** *Let $S$ be a non-empty subset of a finite additive group $Z$. Then we have*

$$\|S\|_{\Lambda(4)} \geq \left(2 - \frac{1}{|S|}\right)^{1/4}, \tag{4.32}$$

*with equality holding if and only if $S$ is a Sidon set. More generally, if $h \geq 1$, then there exists a number $1 \leq \alpha(h, |S|) < (h!)^{1/2h}$ depending on $h$ and $|S|$ such that $\|S\|_{\Lambda(2h)} = \alpha(h, |S|)$ when $S$ is a $B_h$ set, and $\|S\|_{\Lambda(2h)} > \alpha(h, |S|)$ otherwise.*

*Proof* We first prove (4.32). By testing (4.31) with $c_\xi$ identically equal to 1, it will suffice to show that

$$\left\| \sum_{\xi \in S} e(x, \xi) \right\|_{L^4(Z)}^4 \geq \left(2 - \frac{1}{|S|}\right) |S|^2.$$

The left-hand side can be expanded as

$$\sum_{\xi_1, \xi_2, \eta_1, \eta_2 \in S} \mathbf{E}_{x \in Z} e((\xi_1 + \xi_2 - \eta_1 - \eta_2) \cdot x).$$

By Lemma 4.5 this simplifies to

$$|\{\xi_1, \xi_2, \eta_1, \eta_2 \in S : \xi_1 + \xi_2 = \eta_1 + \eta_2\}|.$$

Clearly $\xi_1 + \xi_2$ will equal $\eta_1 + \eta_2$ when $(\xi_1, \xi_2)$ is a permutation of $(\eta_1, \eta_2)$, so this expression is at least as large as

$$\sum_{\xi_1, \xi_2, \eta_1, \eta_2 \in S : \{\xi_1, \xi_2\} = \{\eta_1, \eta_2\}} 1 = |S|(|S| - 1)2 + |S| = \left(2 - \frac{1}{|S|}\right) |S|^2$$

as claimed. Note that this argument also shows that the inequality in (4.32) is strict if $S$ is not a Sidon set, since then we have additional terms coming from pairs $(\xi_1, \xi_2)$ and $(\eta_1, \eta_2)$ which are not permutations of each other.

Now suppose that $S$ is a Sidon set. To prove equality in (4.32) it suffices to show that

$$\left\| \sum_{\xi \in S} c_\xi e(x, \xi) \right\|_{L^4(Z)}^4 \leq 2 - \frac{1}{|S|}$$

assuming the normalization $\sum_{\xi \in S} |c_\xi|^2 = 1$. The left-hand side can be expanded as

$$\sum_{\xi_1, \xi_2, \eta_1, \eta_2 \in S} c_{\xi_1} c_{\xi_2} \overline{c_{\eta_1} c_{\eta_2}} \mathbf{E}_{x \in Z} e((\xi_1 + \xi_2 - \eta_1 - \eta_2) \cdot x)$$

which as before simplifies to

$$\sum_{\xi_1, \xi_2, \eta_1, \eta_2 \in S : \xi_1 + \xi_2 = \eta_1 + \eta_2} c_{\xi_1} c_{\xi_2} \overline{c_{\eta_1} c_{\eta_2}}.$$

Since $S$ is a Sidon set, $(\eta_1, \eta_2)$ must be a permutation of $(\eta_1, \eta_2)$. Splitting into the cases $\xi_1 = \xi_2$ and $\xi_1 \neq \xi_2$, we can thus rewrite the previous expression as

$$\sum_{\xi \in S} \|c_\xi\|_H^4 + 2 \sum_{\xi_1, \xi_2 \in S : \xi_1 \neq \xi_2} |c_{\xi_1}|^2 |c_{\xi_2}|^2$$

which by the normalization $\sum_{\xi \in S} |c_\xi|^2 = 1$ can be written as

$$2 - \sum_{\xi \in S} |c_\xi|^4.$$

But from Cauchy–Schwarz and the normalization $\sum_{\xi \in S} |c_\xi|^2 = 1$ we have $\sum_{\xi \in S} |c_\xi|^4 \geq 1/|S|$, and the claim follows.

The general case $h \geq 2$ is similar but is left to Exercise 4.5.9. □

Another quantification of the heuristic that large $\Lambda(p)$ constants corresponds to strong additive structure is given by

**Lemma 4.30** *Let $S$ be a non-empty subset of a finite additive group $Z$, and let $h \geq 1$. Then we have*

$$|h_1 S - h_2 S| \geq \frac{|S|^h}{\|S\|_{\Lambda(2h)}^{2h}}$$

*whenever $h_1, h_2 \geq 0$ are such that $h_1 + h_2 = h$. In particular we have*

$$|hS| \geq \frac{|S|^h}{\|S\|_{\Lambda(2h)}^{2h}}.$$

**Remark 4.31** This lemma shows that if $S$ has a small $\Lambda(2h)$ constant, then not only do the sum sets $hS$ become very large, but so do the sum sets $hS'$ of all subsets $S'$ of $S$, thanks to the monotonicity of $\Lambda(p)$ constants. The converse statement is also true up to logarithmic factors; see exercises. Thus $\Lambda(2h)$ constants measure the failure of $S$, or any of its subsets, to have good closure properties under $h$-fold sums.

*Proof* From (4.30) with $p := 2h$, and $c_\xi$ set identically equal to 1, we have

$$\left\| \sum_{\xi \in S} e(\xi \cdot x) \right\|_{L^{2h}(Z)}^{2h} \leq \|S\|_{\Lambda(2h)}^{2h} |S|^h.$$

The left-hand side is equal to

$$\left\| \left( \sum_{\xi \in S} e(\xi \cdot x) \right)^{h_1} \left( \sum_{\xi \in -S} e(\xi \cdot x) \right)^{h_2} \right\|_{L^2(Z)}^2$$

since $e(x, -\xi)$ is the conjugate of $e(x, \xi)$. We can expand

$$\left( \sum_{\xi \in S} e(\xi \cdot x) \right)^{h_1} \left( \sum_{\xi \in -S} e(x, \xi) \right)^{h_2} = \sum_{\xi \in S} r_{h_1,h_2}(\xi) e(\xi \cdot x)$$

where $r_{h_1,h_2}$ is the counting function

$$\begin{aligned} r_{h_1,h_2}(\xi) :&= |\{(\xi_1, \ldots, \xi_{h_1}, \xi_1', \ldots, \xi_{h_2}') \in S^{h_1+h_2} : \xi \\ &= \xi_1 + \cdots + \xi_{h_1} - \xi_1' - \cdots - \xi_h'\}|. \end{aligned}$$

By (4.2) we thus have

$$\sum_{\xi \in S} r_{h_1,h_2}(\xi)^2 \leq \|S\|_{\Lambda(2h)}^{2h} |S|^h.$$

On the other hand, the function $r_{h_1,h_2}$ is supported in $h_1 S - h_2 S$, so by Cauchy–Schwarz

$$\sum_{\xi \in S} r_{h_1,h_2}(\xi) \leq |h_1 S - h_2 S|^{1/2} \|S\|_{\Lambda(2h)}^h |S|^{h/2}.$$

But from the definition of $r_{h_1,h_2}$ we have

$$\sum_{\xi \in S} r_{h_1,h_2}(\xi) = |S^{h_1+h_2}| = |S|^{h_1+h_2}$$

The claim follows. □

We now investigate the $\Lambda(p)$ constants of Sidon-like sets as $p \to \infty$.

**Definition 4.32** An additive set $S$ with cardinality $|S| = d$ is said to be *dissociated* if the cube $[0,1]^d \cdot S$ is proper, or in other words, the $2^d$ subset sums

$$FS(S) := \left\{ \sum_{\xi \in S'} \xi : S' \subseteq S \right\}$$

are all distinct.

This should be compared with the concept of a Sidon set, which is a set $S$ of cardinality $d$ whose $\frac{d(d+1)}{2}$ pairwise sums $\{\xi_1 + \xi_2 : \xi_1, \xi_2 \in S\}$ are all distinct (except for the trivial identification $\xi_1 + \xi_2 = \xi_2 + \xi_1$). A good example of a dissociated set is the set of powers of 2: $S = \{1, 2, \dots, 2^n\}$ in any cyclic group $\mathbf{Z}/N\mathbf{Z}$ with $N \geq 2^{n+1}$. Observe that if $S$ is a dissociated set of cardinality $d$, and $v$ is a non-zero element of $[-1, 1]^d$, then $v \cdot S \neq 0$ (since otherwise we could find two disjoint sets $S_1$, $S_2$ in $S$, corresponding to where the components of $v$ are $+1$ or $-1$, such that $\sum_{\xi \in S_1} \xi = \sum_{\xi \in S_2} \xi$).

Dissociativity is the Fourier analog of joint independence. It leads to the following Fourier-analytic analog of Chernoff's inequality:

**Lemma 4.33 (Rudin's inequality)** *If $S$ is dissociated, then we have*

$$\mathbf{E}_{x \in Z} \exp\left(\sigma \mathrm{Re} \sum_{\xi \in S} c(\xi) e(\xi \cdot x)\right) \leq e^{\sigma^2/2} \tag{4.33}$$

*whenever $\|c\|_{l^2(S)} \leq 1$ and $\sigma \geq 0$. We also have the distributional estimates*

$$\mathbf{P}_{x \in Z}\left\{\left|\sum_{\xi \in S} c(\xi) e(\xi \cdot x)\right| \geq \lambda\right\} = O_\varepsilon\big(e^{-\lambda^2/(4+\varepsilon)}\big) \tag{4.34}$$

*for every $\varepsilon > 0$, and the $\Lambda(p)$ estimate*

$$\|S\|_{\Lambda(p)} = O(\sqrt{p}) \tag{4.35}$$

*for all $2 \leq p < \infty$.*

Note that when $p = 2h$ then $(h!)^{1/2h}$ is comparable to $\sqrt{p}$ by Stirling's formula (1.52), and hence so (4.35) and shows that dissociated sets are comparable in $\Lambda(2h)$ constant to $B_{2h}$ sets for any given $h$ (if $S$ is sufficiently large). This also shows that the bounds in the above lemma cannot be significantly improved except in the constants, even if one imposes even more additive independence conditions on $S$.

*Proof* Write $c(\xi) = |c(\xi)| e(\theta_\xi)$ for some phase $\theta_\xi \in \mathbf{R}/\mathbf{Z}$. We begin by observing the inequality

$$e^{tx} \leq \cosh(x) + t \sinh(x)$$

for all $x \geq 0$ and $-1 \leq t \leq 1$, which is simply a consequence of the convexity of $e^{tx}$ as a function of $t$. In particular we see that

$$\exp(\sigma \mathrm{Re} c(\xi) e(x, \xi)) \leq \cosh(\sigma |c(\xi)|) + \sinh(\sigma |c(\xi)|) \mathrm{Re}\, e(\xi \cdot x + \theta_\xi),$$

which upon multiplying and taking expectations becomes

$$\begin{aligned}&\mathbf{E}_{x\in Z}\exp\left(\sigma\sum_{\xi\in S}\mathrm{Re}c(\xi)e(x,\xi)\right)\\&\quad\leq\mathbf{E}_{x\in Z}\prod_{\xi\in S}\left((\cosh(\sigma|c(\xi)|)+\frac{1}{2}\sinh(\sigma|c(\xi)|)e(\xi\cdot x+\theta_\xi)\right.\\&\qquad\qquad\left.+\frac{1}{2}\sinh(\sigma|c(\xi)|)e(-\xi\cdot x-\theta_\xi)\right).\end{aligned}$$

Now we multiply the product out and inspect its behavior in $x$. We obtain a large number of terms ($3^{|S|}$, to be exact) that are of the form $e((v\cdot S)\cdot\xi)$, for some $v\in[-1,1]^{|S|}$, times some constant independent of $x$, where we select some enumeration $S=(\xi_1,\ldots,\xi_{|S|})$ of $S$. There is one constant term, namely $\prod_{\xi\in S}\cosh(\sigma|c(\xi)|)$, but all the others have a non-zero frequency vector $v\cdot S$ because $S$ is dissociated, and thus integrate out to zero by the Fourier inversion formula. Thus we have

$$\mathbf{E}_{x\in Z}\exp\left(\sigma\sum_{\xi\in S}\mathrm{Re}\,c(\xi)e(\xi\cdot x)\right)\leq\prod_{\xi\in S}\cosh(\sigma|c(\xi)|),$$

and the claim (4.33) then follows from the elementary inequality $\cosh(x)\leq e^{x^2/2}$ (which follows by comparing Taylor series). From Markov's inequality we thus obtain

$$\mathbf{P}_{x\in Z}\left(\mathrm{Re}\sum_{\xi\in S}c(\xi)e(\xi\cdot x)\geq\lambda\right)\leq e^{\sigma^2/\lambda}e^{-\sigma\lambda}$$

for every $\lambda\geq 0$; choosing $\sigma:=\lambda/2$, we obtain

$$\mathbf{P}_{x\in Z}\left(\mathrm{Re}\sum_{\xi\in S}c(\xi)e(\xi\cdot x)\geq\lambda\right)\leq e^{-\lambda^2/4}.$$

Replacing $\lambda$ by $(1-\varepsilon)\lambda$ and rotating $c(\xi)$ by an arbitrary angle $e(\theta)$, we obtain

$$\mathbf{P}_{x\in Z}\left(\mathrm{Re}\,e(\theta)\sum_{\xi\in S}c(\xi)e(\xi\cdot x)\geq(1-\varepsilon)\lambda\right)\leq e^{-\lambda^2(1-\varepsilon^2)/4}.$$

If take the union of these estimates as $e^{i\theta}$ varies over a finite number of angles (depending on $\varepsilon$) we obtain (4.34).

To obtain (4.35), we observe from the identity

$$\left\|\sum_{\xi\in S}c(\xi)e(\xi\cdot x)\right\|_{L^p(Z)}^p\leq p\int_0^\infty\lambda^{p-1}\mathbf{P}_{x\in Z}\left(\left|\sum_{\xi\in S}c(\xi)e(\xi\cdot x)\right|\geq\lambda\right)\,d\lambda$$

and (4.34) (with $\varepsilon = 1$, say) that

$$\left\| \sum_{\xi \in S} c(\xi) e(\xi \cdot x) \right\|_{L^p(Z)}^p = O\left( p \int_0^\infty \lambda^{p-1} e^{-\lambda^2/5} \, d\lambda \right).$$

To estimate the integral, we observe from elementary calculus that the integrand $\lambda^{p-1} e^{-\lambda^2/5}$ is bounded by $O(p)^{p/2}$ for $\lambda = O(\sqrt{p})$, and then decays exponentially for $\lambda \gg \sqrt{p}$. From this we can easily bound the integrand by $p^{O(1)} O(p)^{p/2}$, and the claim follows (note that $p^{1/p}$ is bounded by $e$). $\square$

In the next few sections we shall use Rudin's inequality to obtain structural control on various sets of frequencies.

## Exercises

4.5.1 Show that the $\Lambda(p)$ constant of a set $S$ does not depend on the choice of bilinear form used to define the Fourier transform, and is also invariant under translations or isomorphisms of the set $S$.

4.5.2 For any $2 \leq p \leq \infty$ and any disjoint $S_1, S_2$, show the triangle inequality $\|S\|_{\Lambda(p)} \leq \|S_1\|_{\Lambda(p)} + \|S_2\|_{\Lambda(p)}$ whenever $S \subseteq S_1 \cup S_2$.

4.5.3 Let $\varepsilon$ be the uniform distribution on $\{-1, 1\}$, and let $\varepsilon_1, \ldots, \varepsilon_N$ be independent trials of $\varepsilon$. If $c_1, \ldots, c_N$ are arbitrary complex numbers and $2 \leq p < \infty$, prove *Bernstein's inequality* [25]

$$\begin{aligned} \left( \sum_{j=1}^N |c_j|^2 \right)^{1/2} &\leq \mathbf{E} \left( \left| \sum_{j=1}^N \varepsilon_j c_j \right|^p \right)^{1/p} \\ &\leq O\left( \sqrt{p} \left( \sum_{j=1}^N |c_j|^2 \right)^{1/2} \right). \end{aligned}$$

(Hint: for the lower bound, compute the $p = 2$ moment. For the upper bound, modify the proof of Lemma 4.33; alternatively, apply Lemma 4.33 to the group $Z = \mathbf{Z}_2^N$, where $S$ is the standard basis for $\mathbf{Z}_2^N$.) Conclude that if $f_1, \ldots, f_N$ are any complex-valued functions on $Z$, then we have *Khintchine's inequality*

$$\begin{aligned} \left\| \left( \sum_{j=1}^N |f_j|^2 \right)^{1/2} \right\|_{L^p(Z)} &\leq \mathbf{E} \left( \left\| \sum_{j=1}^N \varepsilon_j f_j \right\|_{L^p(Z)}^p \right)^{1/p} \\ &\leq O\left( \sqrt{p} \left\| \left( \sum_{j=1}^N |f_j|^2 \right)^{1/2} \right\|_{L^p(Z)} \right). \end{aligned}$$

4.5.4 Let $f : Z_1 \times Z_2 \to \mathbf{C}$ be a function on two variables in two non-empty finite sets $Z_1, Z_2$, and let $2 \leq p < \infty$. Establish the Minkowski inequality

$$\big(\mathbf{E}_{y \in Z_2}(\mathbf{E}_{x \in Z_1}|f(x,y)|^2)^{p/2}\big)^{1/p} \leq \big(\mathbf{E}_{x \in Z_1}(\mathbf{E}_{y \in Z_2}|f(x,y)|^p)^{2/p}\big)^{1/2} \tag{4.36}$$

(Hint: use the triangle inequality for the $L^{p/2}$ norm.) Conclude that $\|S\|_{\Lambda(p)}$ is the best constant such that

$$\left\| \left\| \sum_{\xi \in S} c(\xi) e(x, \xi) \right\|_H \right\|_{L^p(Z)} \leq \|S\|_{\Lambda(p)} \left( \sum_{\xi \in S} \|c(\xi)\|_H^2 \right)^{1/2}$$

for all finite-dimensional Hilbert spaces $H$ and all sequences $(c(\xi))_{\xi \in S}$ taking values in $H$. Using this, conclude that $\|S_1 \times S_2\|_{\Lambda(p)} = \|S_1\|_{\Lambda(p)} \|S_2\|_{\Lambda(p)}$ whenever $S_1, S_2$ are additive sets in finite additive groups $Z_1, Z_2$ and $2 \leq p \leq \infty$.

4.5.5 [33], [20] Let $n \geq 1$ be an integer, let $Z := \mathbf{Z}_2^n$. For $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{Z}_2^n$, let $|\xi|$ denote the number of coefficients $\xi_1, \ldots, \xi_n$ which are equal to one. Establish the *Bonami–Beckner inequality*

$$\left\| \sum_{\xi \in Z} \varepsilon^{|\xi|} c(\xi) \right\|_{L^{1+1/\varepsilon^2}(Z)} \leq \|c\|_{l^2(Z)}$$

for all $0 < \varepsilon < 1$ and all $c \in l^2(\mathrm{Z})$. (Hint: first establish this by hand for $n = 1$, and then exploit (4.36) to obtain the general case.) Conclude in particular that if $S_k := \{\xi \in \mathbf{Z}_2^n : |\xi| = k\}$, then $\|S_k\|_{\Lambda(p)} \leq (p-1)^{k/2}$ for all $2 < p < \infty$.

4.5.6 Let $2 \leq p \leq \infty$, and let $S$ be a non-empty subset of $Z$. Prove (4.31). (Hint: use the Hausdorff–Young inequality.) If $2 < p < \infty$, show that equality occurs if and only if $S$ is a translate of a subgroup of $Z$. (You may need Exercise 4.2.9.)

4.5.7 Let $S$ be an additive set in a finite additive group. Show that

$$\|S\|_{\Lambda(p)} \geq \min\big(1, |Z|^{-1/p}|S|^{1/2}\big)$$

for all $2 \leq p < \infty$. It turns out that these bounds are essentially sharp for randomly chosen sets $S$ in $Z$ of a fixed cardinality: see [35].

4.5.8 Let $S$ be a $B_h$ set in a finite additive group $Z$. Show that $|S| \leq |Z|^{1/h}$.

4.5.9 Complete the proof of Proposition 4.29.

4.5.10 Let $S$ be an additive subset of $Z$. Show that $E(S,S) \leq \|S\|_{\Lambda(4)}^4 |S|^2$; thus the additive energy of an additive set is controlled by its $\Lambda(4)$ constant.

4.5.11 Let $S$ be an additive set, and let $h \geq 1$. Suppose that $A > 0$ is a constant such that

$$|hS'| \geq \frac{|S'|^h}{A^{2h}}$$

for all non-empty subsets $S'$ of $S$. Show that

$$\|S\|_{\Lambda(2h)} = O(A(1 + \log |S|));$$

thus Lemma 4.30 can be reversed after conceding a factor of a logarithm. (Hint: first verify the estimate (4.30) when $c$ is a characteristic function by reversing the proof of Lemma 4.30. For general $c$, decompose $c$ into at most $O(1 + \log |S|)$ functions which are comparable to constant multiples of characteristic functions, by partitioning the range of $c$ using powers of 2, and discarding those values of $c$ smaller than (say) $|S|^{-100}\|c\|_{l^2}$.)

4.5.12 [251] Show that $\|S\|_{\Lambda(p)}$ is the best constant such that

$$\|\hat{f}\|_{l^2(S)} \leq \|S\|_{\Lambda(p)} \|f\|_{L^{p'}(Z)}$$

for all random variables $f$, where $p'$ is the dual exponent to $p$, thus $1/p + 1/p' = 1$. Next, write

$$\|\hat{f}\|^2_{l^2(Z)} = \frac{|S|}{|Z|}\|f\|^2_{L^2(Z)} + \mathbf{E}_{x,y\in Z} f(x)\overline{f(y)}\mathbf{I}(x \neq y)\sum_{\xi\in S} e(\xi \cdot (x - y))$$

and observe the inequalities

$$\left|\mathbf{E}_{x,y\in Z} f(x)g(y)\mathbf{I}(x \neq y)\sum_{\xi\in S} e(\xi \cdot (x - y))\right| \leq \|f\|_{L^2(Z)}\|g\|_{L^2(Z)}$$

and

$$\left|\mathbf{E}_{x,y\in Z} f(x)g(y)\mathbf{I}(x \neq y)\sum_{\xi\in S} e(\xi \cdot (x - y))\right| \leq |Z|\|S\|_u\|f\|_{L^1(Z)}\|g\|_{L^1(Z)}.$$

Using Riesz–Thorin interpolation (or arguing as in Exercise 4.2.3) conclude that

$$\left|\mathbf{E}_{x,y\in Z} f(x)g(y)\mathbf{I}(x \neq y)\sum_{\xi\in S} e(\xi \cdot (x - y))\right| \\ \leq (|Z|\|S\|_u)^{1-2/p}\|f\|_{L^{p'}(Z)}\|g\|_{L^{p'}(Z)}.$$

From this, conclude the *Tomas–Stein inequality*

$$\|S\|^2_{\Lambda(p)} \leq |S||Z|^{-\frac{2}{p}} + (\|S\|_u|Z|)^{1-\frac{2}{p}}$$

(compare with (4.31)). Thus, Fourier-uniform sets tend to have fairly small $\Lambda(p)$ constants. See also Lemma 10.22.