

4.6 The spectrum of an additive set

We now use Fourier analysis to investigate the spectral properties of additive sets A which have high additive energy $E(A, A)$; examples of such sets include sets with small sum set $|A + A|$ or small difference set $|A - A|$ (cf. (2.8)). One can already conclude from estimates such as (4.23) that such sets must be highly non-uniform, i.e. 1_A contains non-trivial Fourier coefficients. However, this by itself is not the strongest Fourier-analytic statement one can say about such sets. In order to proceed further it is convenient to introduce the notion of the α -spectrum of a set.

Definition 4.34 (Spectrum) Let A be an additive set in a finite additive group Z with a non-degenerate symmetric bilinear form \cdot and let $\alpha \in \mathbf{R}$ be a parameter. We define the α -spectrum $\text{Spec}_\alpha(A) \subseteq Z$ to be the set

$$\text{Spec}_\alpha(A) := \{\xi \in Z : |\widehat{1_A}(\xi)| \geq \alpha \mathbf{P}_Z(A)\}.$$

One could define this spectrum without the assistance of the bilinear form \cdot , but then it would be a subset of the Pontryagin dual group \widehat{Z} rather than Z .

From Lemma 4.9 we see that the sets $\text{Spec}_\alpha(A)$ are symmetric, decreasing in α , empty for $\alpha > 1$, contain the origin for $\alpha \leq 1$, and are the whole space Z whenever $\alpha \leq 0$. Thus the spectrum is really only an interesting concept when $0 < \alpha \leq 1$. In the extreme case $\alpha = 1$ the spectrum becomes a group, see Exercise 4.6.2.

From (4.16) (and Markov's inequality) we observe the upper bound

$$|\text{Spec}_\alpha(A)| \leq \alpha^{-2} / \mathbf{P}_Z(A) \tag{4.37}$$

on the cardinality of the α -spectrum. In fact we can use Rudin's inequality to obtain a more precise structural statement, in which the polynomial loss in $\mathbf{P}_Z(A)$ is replaced with a logarithmic loss. To prove this statement, we first need an easy lemma (cf. Corollary 1.42).

Lemma 4.35 (Cube covering lemma) [36] *Let S be an additive set in an ambient group Z , and let $d \geq 1$ be an integer. Then we can partition $S = D_1 \cup \dots \cup D_k \cup R$ where D_1, \dots, D_k are disjoint dissociated subsets of S of cardinality $d + 1$, and the remainder set R is contained in a cube $[-1, 1]^d \cdot (\eta_1, \dots, \eta_d)$ for some $\eta_1, \dots, \eta_d \in Z$.*

Proof We use the greedy algorithm. We initially set $k = 0$. If we can find a dissociated subset D of S of cardinality $d + 1$, we remove it from S and add it to the collection D_1, \dots, D_k , thus incrementing $k + 1$. We continue in this manner until we are left with a remainder R where all dissociated subsets of S have cardinality d or less. Let $\{\eta_1, \dots, \eta_{d'}\}$ be a dissociated subset of R with maximal cardinality; thus $d' \leq d$. Observe that if R contained an element ξ which was not contained in $[-1, 1]^{d'} \cdot (\eta_1, \dots, \eta_{d'})$, then $\{\eta_1, \dots, \eta_{d'}, \xi\}$ would be dissociated,

so contradicting maximality of d' . Thus we have $R \subseteq [-1, 1]^{d'} \cdot (\eta_1, \dots, \eta_{d'})$, and the claim follows (padding out the progression with some dummy elements $\eta_{d'+1}, \dots, \eta_d$ if necessary). \square

Lemma 4.36 (Fourier concentration lemma) [48] *Let A be an additive set in a finite additive group Z , and let $0 < \alpha \leq 1$. Then there exist $d = O(\alpha^{-2}(1 + \log \frac{1}{\mathbf{P}_Z(A)}))$ and frequencies $\eta_1, \dots, \eta_d \in Z$ such that*

$$\text{Spec}_\alpha(A) \subseteq [-1, 1]^d \cdot (\eta_1, \dots, \eta_d).$$

This result is essentially sharp in a number of ways; see [146].

Proof It will suffice to show that for each phase $\theta \in \mathbf{R}/\mathbf{Z}$, the set

$$S_\theta := \left\{ \xi \in Z : \text{Re } e(\theta) \widehat{1}_A(\xi) \geq \frac{\alpha}{2} \mathbf{P}_Z(A) \right\}$$

can be contained in a progression of the desired form, since from Definition 4.34 we see that $\text{Spec}_\alpha(A)$ is contained in the union of a bounded number of the S_θ , and we can simply add all the progressions together (here the fact that we have $\alpha/2$ instead of α in the definition of S_θ is critical).

Fix θ . By Lemma 4.35, it will suffice to show that

$$|S'| \leq C\alpha^{-2} \left(1 + \log \frac{1}{\mathbf{P}_Z(A)} \right)$$

for all dissociated sets S' in S_θ . But if $S' \in S_\theta$, then by definition of S_θ

$$\text{Re } e(\theta) \sum_{\xi \in Z} \widehat{1}_A(\xi) 1_{S'}(\xi) \geq \frac{\alpha}{2} \mathbf{P}_Z(A) |S'|.$$

Let $f(x) := \frac{1}{|S'|^{1/2}} \sum_{\xi \in S'} e(x, \xi)$ be the normalized inverse Fourier transform of $1_{S'}$; then by (4.3) the left-hand side is equal to $\text{Re } e(\theta) |S'|^{1/2} \mathbf{E}_Z 1_A f$. Thus we have

$$\mathbf{E}_Z 1_A |f| \geq \frac{\alpha}{2} \mathbf{P}_Z(A) |S'|^{1/2}.$$

The left-hand side can be rewritten as

$$\mathbf{E}_Z 1_A |f| = \int_0^\infty \mathbf{P}_{x \in Z}(x \in A; |f(x)| \geq \lambda) d\lambda,$$

cf. (1.6). To bound $\mathbf{P}_{x \in Z}(x \in A; |f(x)| \geq \lambda)$, we can either use the trivial bound of $\mathbf{P}_Z(A)$ or use (4.34) to obtain a bound of $Ce^{-\lambda^2/5}$ (for instance). Thus we have

$$\int_0^\lambda \min(\mathbf{P}_Z(A), Ce^{-\lambda^2/5}) d\lambda \geq \frac{\alpha}{2} \mathbf{P}_Z(A) |S'|^{1/2}.$$

The left-hand side is at most $C\mathbf{P}_Z(A)(1 + \log^{1/2} \frac{1}{\mathbf{P}_Z(A)})$, and the claim follows. \square

The above lemma suggests that the spectrum has some additive structure. This is confirmed by the following closure properties of the α -spectrum under addition:

Lemma 4.37 *Let A be an additive set in an finite additive group Z , and let $\varepsilon, \varepsilon' > 0$. Then we have*

$$\text{Spec}_{1-\varepsilon}(A) + \text{Spec}_{1-\varepsilon'}(A) \subseteq \text{Spec}_{1-2(\varepsilon+\varepsilon')}. \quad (4.38)$$

In a similar spirit, for any $0 < \alpha \leq 1$ and for any non-empty $S \subseteq \text{Spec}_\alpha(A)$ we have

$$\left| \{(\xi_1, \xi_2) \in S \times S : \xi_1 - \xi_2 \in \text{Spec}_{\alpha^2/2}(A)\} \right| \geq \frac{\alpha^2}{2} |S|^2 \quad (4.39)$$

See Exercise 4.6.2 for the $\varepsilon = 0$ case of this lemma. This lemma should be compared with Lemma 2.33. Indeed there is a strong analogy between the spectra $\text{Spec}_\alpha(A)$ and the symmetry sets $\text{Sym}_\alpha(A)$, which are heuristically dual to each other.

Proof We first prove (4.38). Let $\xi \in \text{Spec}_{1-\varepsilon}$ and $\xi' \in \text{Spec}_{1-\varepsilon'}$, then there exists phases $\theta, \theta' \in \mathbf{R}/\mathbf{Z}$ such that

$$\begin{aligned} \text{Re } \mathbf{E}_{x \in Z} e(\xi \cdot x + \theta) 1_A(x) &\geq (1 - \varepsilon) \mathbf{P}_Z(A); \\ \text{Re } \mathbf{E}_{x \in Z} e(\xi' \cdot x + \theta') 1_A(x) &\geq (1 - \varepsilon') \mathbf{P}_Z(A). \end{aligned}$$

Since $\text{Re } \mathbf{E}_{x \in Z} 1_A = \mathbf{P}_Z(A)$, we thus have

$$\text{Re } \mathbf{E}_{x \in Z} [2e(\xi \cdot x + \theta) + 2e(\xi' \cdot x + \theta') - 3] 1_A(x) \geq (1 - 2(\varepsilon + \varepsilon')) \mathbf{P}_Z(A).$$

To conclude that $\xi + \xi' \in \text{Spec}_{1-2(\varepsilon+\varepsilon')}(A)$, it will thus suffice to establish the pointwise estimate

$$\text{Re } [2e(\xi \cdot x + \theta) + 2e(\xi' \cdot x + \theta') - 3] \leq \text{Re } [e^{i(\theta+\theta')} e(x, \xi + \xi')].$$

Writing $e(\xi \cdot x + \theta) = e^{i\beta}$ and $e(\xi' \cdot x + \theta') = e^{i\beta'}$ for some $-\pi/2 \leq \beta, \beta' \leq \pi/2$, we reduce to showing

$$2 \cos(\beta) + 2 \cos(\beta') - 3 \geq \cos(\beta + \beta').$$

But by the convexity of \cos between $-\pi/2$ and $\pi/2$, we have

$$\begin{aligned} 2 \cos(\beta) + 2 \cos(\beta') - 3 &\geq 4 \cos\left(\frac{\beta + \beta'}{2}\right) - 3 \\ &= 2 \cos\left(\frac{\beta + \beta'}{2}\right)^2 - 1 - 2 \left(1 - \cos\left(\frac{\beta + \beta'}{2}\right)\right)^2 \\ &\geq \cos(\beta + \beta') \end{aligned}$$

as desired.

Now we prove (4.39), which is due to Bourgain [41]. Set $a(\xi) := \text{sgn}(\widehat{1}_A(\xi))$ for $\xi \in S$; thus

$$\mathbf{E}_{x \in Z} \sum_{\xi \in S} a(\xi) e(\xi \cdot x) 1_A(x) = \sum_{\xi \in S} |\widehat{1}_A(\xi)| \geq \alpha \mathbf{P}_Z(A) |S|.$$

Applying Cauchy–Schwarz, we conclude

$$\mathbf{E}_{x \in Z} \left| \sum_{\xi \in S} a(\xi) e(\xi \cdot x) \right|_{1_A(x)}^2 \geq \alpha^2 \mathbf{P}_Z(A) |S|^2.$$

But the left-hand side can be rearranged as

$$\sum_{\xi_1, \xi_2 \in S} a(\xi_1) \overline{a(\xi_2)} \widehat{1}_A(\xi_1 - \xi_2),$$

so by the triangle inequality we have

$$\sum_{\xi_1, \xi_2 \in S} |\widehat{1}_A(\xi_1 - \xi_2)| \geq \alpha^2 |S|^2.$$

In particular (cf. Exercise 1.1.4)

$$\sum_{\xi_1, \xi_2 \in S: \xi_1 - \xi_2 \in \text{Spec}_{\alpha^2/2}(A)} |\widehat{1}_A(\xi_1 - \xi_2)| \geq \alpha^2/2 |S|^2$$

and (4.39) follows. □

We now show that small sum sets force large spectra (cf. Exercise 4.3.9, or Exercise 4.6.3 below).

Lemma 4.38 *Let A be an additive set in an finite additive group Z , and let $0 < \alpha \leq 1$. For any integers $n, m \geq 0$ with $(n, m) \neq (0, 0)$, we have the lower bound on sum sets*

$$|nA - mA| \geq \frac{|A|}{|\text{Spec}_{\alpha}(A) \mathbf{P}_Z(A) + \alpha^{2(n+m)-2}|}.$$

Proof We may take $n, m \geq 0$. Consider the function $f = 1_A * \dots * 1_A * 1_{-A} * \dots * 1_{-A}$ formed by convolving n copies of A and m copies of $-A$. Then f is non-negative and supported on $nA - mA$, and thus

$$\mathbf{E}_Z f \leq \mathbf{P}_Z(nA - mA)^{1/2} (\mathbf{E}_Z |f|^2)^{1/2}.$$

From (4.10) we have $\mathbf{E}_Z f = \mathbf{P}_Z(A)^{n+m}$. From (4.9) and (4.17) we have $\widehat{f} = \widehat{1}_A^n \widehat{1}_{-A}^m$. Combining these inequalities with (4.2) we see that

$$|nA - mA| \geq \frac{|Z| \mathbf{P}_Z(A)^{2(n+m)}}{\sum_{\xi \in Z} |\widehat{1}_A(\xi)|^{2(n+m)}}.$$

But

$$\begin{aligned} \sum_{\xi \in Z} |\widehat{1}_A(\xi)|^{2(n+m)} &\leq \sum_{\xi \in \text{Spec}_\alpha(A)} \mathbf{P}_Z(A)^{2(n+m)} \\ &\quad + \sum_{\xi \notin \text{Spec}_\alpha(A)} \alpha^{2(n+m)-2} \mathbf{P}_Z(A)^{2(n+m)-2} |\widehat{1}_A(\xi)|^2 \\ &\leq \mathbf{P}_Z(A)^{2(n+m)} |\text{Spec}_\alpha(A)| + \alpha^{2(n+m)-2} \mathbf{P}_Z(A)^{2(n+m)-1} \end{aligned}$$

and the claim follows. \square

Now we consider the following inverse-type question: if A has additive structure in the sense that its energy $E(A, A)$ is large or its difference set $|A - A|$ is small, is it possible to approximate A (or a closely related set) by a Bohr set? We give two results of this type, one which places a relatively large Bohr set inside $2A - 2A$, and another which places $A - A$ inside a relatively small Bohr set. We begin with the former result, the main idea of which dates back to Bogolyubov.

Proposition 4.39 [295] *Let $0 < \alpha \leq 1$, and let A be an additive set in a finite additive group Z such that $E(A, A) \geq 4\alpha^2|A|^3$. Then we have the inclusion*

$$\text{Bohr}\left(\text{Spec}_\alpha(A), \frac{1}{6}\right) \subseteq 2A - 2A. \quad (4.40)$$

Proof Let x be any element of the Bohr set $\text{Bohr}(\text{Spec}_\alpha(A), \frac{1}{6})$, thus $\text{Re } e(\xi \cdot x) > \frac{1}{2}$ for all $\xi \in \text{Spec}_\alpha(A)$. To show that $x \in 2A - 2A$, it would suffice to show that $1_A * 1_A * 1_{-A} * 1_{-A}(x) \neq 0$. But from (4.4), (4.9), (4.17) we have

$$1_A * 1_A * 1_{-A} * 1_{-A}(x) = \sum_{\xi \in Z} |\widehat{1}_A(\xi)|^4 e(\xi \cdot x).$$

Now take real parts of both sides and use the hypothesis on x to obtain

$$\begin{aligned} 1_A * 1_A * 1_{-A} * 1_{-A}(x) &= \sum_{\xi \in \text{Spec}_\alpha(A)} |\widehat{1}_A(\xi)|^4 \text{Re } e(\xi \cdot x) + \sum_{\xi \notin \text{Spec}_\alpha(A)} |\widehat{1}_A(\xi)|^4 \text{Re } e(x, \xi) \\ &\geq \frac{1}{2} \sum_{\xi \in \text{Spec}_\alpha(A)} |\widehat{1}_A(\xi)|^4 - \sum_{\xi \notin \text{Spec}_\alpha(A)} |\widehat{1}_A(\xi)|^4 \\ &= \frac{1}{2} \sum_{\xi \in Z} |\widehat{1}_A(\xi)|^4 - \frac{3}{2} \sum_{\xi \notin \text{Spec}_\alpha(A)} |\widehat{1}_A(\xi)|^4 \\ &\geq \frac{1}{2} \frac{E(A, A)}{|Z|^3} - \frac{3}{2} \sum_{\xi \in Z} \alpha^2 \mathbf{P}_Z(A)^2 |\widehat{1}_A(\xi)|^2 \\ &\geq \frac{1}{2} \frac{E(A, A)}{|Z|^3} - \frac{3}{2} \alpha^2 \mathbf{P}_Z(A)^3 \\ &> 0 \end{aligned}$$

as desired, where we have used the hypothesis on α in the last step. \square

Now we give a converse inclusion, which applies to sets of small difference constant $\delta[A]$ but requires the spectral threshold to be very large.

Proposition 4.40 *Let $K \geq 1$. If A is an additive set in a finite additive group Z such that $|A - A| \leq K|A|$ (i.e. $\delta[A] \leq K$) and $0 < \varepsilon < 1$, then*

$$A - A \subseteq \text{Bohr}(\text{Spec}_{1-\varepsilon}(A - A), \sqrt{8\varepsilon K}).$$

Proof Let $x, y \in A$ and $\xi \in \text{Spec}_{1-\varepsilon}(A - A)$. Then there exists a phase $\theta \in \mathbf{R}/\mathbf{Z}$ such that

$$\text{Re} \sum_{z \in A-A} e(\xi \cdot x + \theta) \geq (1 - \varepsilon)|A - A|$$

and hence

$$\sum_{z \in A-A} (1 - \text{Re} e(\xi \cdot x + \theta)) \leq \varepsilon|A - A| \leq \varepsilon K|A|.$$

Since the summand is non-negative, and $A - A$ contains both $x - a$ and $y - a$, we thus have

$$\sum_{a \in A} |1 - \text{Re} e(\xi \cdot (x - a) + \theta)| \leq \varepsilon K|A|$$

and hence by Cauchy–Schwarz

$$\sum_{a \in A} |1 - \text{Re} e(\xi \cdot (x - a) + \theta)|^{1/2} \leq \varepsilon^{1/2} K^{1/2} |A|.$$

From the elementary identity

$$|1 - e(\alpha)| = \sqrt{2}|1 - \text{Re} e(\alpha)|^{1/2}$$

we conclude that

$$\sum_{a \in A} |1 - e(\xi \cdot (x - a) + \theta)| \leq \sqrt{2}\varepsilon^{1/2} K^{1/2} |A|.$$

Similarly for x replaced by y . By the triangle inequality we conclude that

$$\sum_{a \in A} |e(\xi \cdot (y - a) + \theta) - e(\xi \cdot (x - a) + \theta)| \leq \sqrt{2}2\varepsilon^{1/2} K^{1/2} |A|.$$

But the left-hand side is just $|A|e(\xi \cdot (x - y))$; thus

$$|e(\xi \cdot (x - y)) - 1| \leq \sqrt{8\varepsilon K}.$$

Since $\xi \in \text{Spec}_{1-\varepsilon}(A - A)$ was arbitrary, the claim follows from (4.24). \square

In the next chapter we apply these propositions, together with the additive geometry results from Chapter 3, to obtain Freiman-type theorems in finite additive

groups. For now, we shall give one striking application of the above machinery, namely the following Gauss sum estimate of Bourgain and Konyagin:

Theorem 4.41 [44] *Let $F = F_p$ be a finite field of prime order, and let H be a multiplicative subgroup of F such that $|H| \geq p^\delta$ for some $0 < \delta < 1$. Then, if p is sufficiently large depending on δ , we have $\|H\|_u \leq p^{-\varepsilon}$ for some $\varepsilon = \varepsilon(\delta) > 0$. In other words, we have*

$$\sup_{\xi \in \mathbf{Z}_p \setminus 0} \left| \sum_{x \in H} e(x\xi) \right| \leq p^{-\varepsilon} |H|.$$

Proof We may use the standard bilinear form $\xi \cdot x = x\xi/p$. Since $h \cdot H = H$ for all $h \in H$, we easily verify that $\hat{1}_H(h^{-1}\xi) = \hat{1}_H(\xi)$ for all $h \in H$ and $\xi \in \mathbf{Z}$. This implies in particular that $\text{Spec}_\alpha(H) = H \cdot \text{Spec}_\alpha(H)$. Thus each $\text{Spec}_\alpha(H)$ consists of multiplicative cosets of H , together with the origin 0.

We use an iteration and pigeonhole argument, similar to that used to prove Theorem 2.35. Let $J = J(\delta) \geq 1$ be a large integer to be chosen later, and let $\varepsilon = \varepsilon(J, \delta) > 0$ be a small number also to be chosen later. Define the sequence $1 > \alpha_1 > \dots > \alpha_{J+1} > 0$ by setting $\alpha_1 := p^{-\varepsilon}$ and $\alpha_{j+1} := \alpha_j^2/2$. Suppose for contradiction that $\|H\|_u > p^{-\varepsilon}$; then $\text{Spec}_{\alpha_1}(H)$ contains a non-zero element, and hence by the preceding discussion $|\text{Spec}_{\alpha_1}(H)| \geq |H| + 1 \geq p^\delta + 1$. Since $\text{Spec}_{\alpha_j}(H)$ is increasing in j , we see from the pigeonhole principle that there exists $1 \leq j \leq J$ such that

$$|\text{Spec}_{\alpha_{j+1}}(H)| \leq p^{1/J} |\text{Spec}_{\alpha_j}(H)|.$$

On the other hand, from Lemma 4.37 we have

$$|\{(\xi_1, \xi_2) \in \text{Spec}_{\alpha_j}(H) \times \text{Spec}_{\alpha_j}(H) : \xi_1 - \xi_2 \in \text{Spec}_{\alpha_{j+1}}(A)\}| \geq \frac{\alpha_j^2}{2} |\text{Spec}_{\alpha_j}(H)|^2.$$

Applying Cauchy–Schwarz or Lemma 2.30 we conclude that

$$E(\text{Spec}_{\alpha_j}(H), \text{Spec}_{\alpha_j}(H)) = \Omega_J(p^{-O_J(\varepsilon) - O(1/J)} |\text{Spec}_{\alpha_j}(H)|^3).$$

If we let $A := \text{Spec}_{\alpha_j}(H) \setminus \{0\}$, we thus obtain

$$E(A, A) = \Omega_J(p^{-O_J(\varepsilon) - O(1/J)} |A|^3)$$

since $|A| \geq p^\delta$, J is large enough depending on δ , and ε small enough depending on J, δ . But A is a union of cosets $x \cdot H$ of H for various $x \in F_p \setminus \{0\}$. Applying Exercise 2.3.20

$$E(A, x \cdot H) = \Omega_J(p^{-O_J(\varepsilon) - O(1/J)} |A| |H|^2).$$

Dilating this by x^{-1} we obtain

$$E(x^{-1} \cdot A, H) = \Omega_J(p^{-O_J(\varepsilon) - O(1/J)} |A| |H|^2).$$

But this will contradict Corollary 2.62 if J is sufficiently large depending on δ , and ε sufficiently small. \square

In [40] this result was extended (using slightly different arguments) to the case where H was not a multiplicative subgroup, but merely had small multiplicative doubling, for instance $|H \cdot H| \leq p^\varepsilon |H|$. In [41] the result was further extended to the case where the field F_p was replaced by a commutative ring such as $F_p \times F_p$ (with Theorem 2.63 playing a key role in the latter result). This yields some estimates on exponential sums related to the Diffie–Hellman distribution and to Mordell sums; see [40], [41] for further discussion.

Exercises

- 4.6.1 Let A be an additive set in a finite additive group Z and let $\alpha \in \mathbf{R}$. Show that A , $-A$, and $T^h A$ all have the same spectrum for any $h \in Z$; thus $\text{Spec}_\alpha(A) = \text{Spec}_\alpha(-A) = \text{Spec}(T^h A)$. If $\phi : Z \rightarrow Z$ is a group isomorphism of Z , show that $\text{Spec}_\alpha(\phi(A)) = \phi^\dagger(\text{Spec}_\alpha(A))$, where ϕ^\dagger is the adjoint of ϕ , defined in Exercise 4.1.8.
- 4.6.2 Let A be an additive set in Z . Show that the spectrum $\text{Spec}_1(A)$ is a group and is in fact equal to $(A - A)^\perp$, the orthogonal complement of the group generated by $A - A$. Also, recall that $\text{Sym}_0(A) := \{h \in A : A + h = A\}$ is the symmetry group of A ; show that the orthogonal complement $\text{Sym}_0(A)^\perp$ of this group is the smallest group which contains the $\text{Spec}_\alpha(A)$ for all $\alpha > 0$.
- 4.6.3 Let A be an additive set in an finite additive group Z , and let $0 < \alpha \leq 1$. Establish the inequalities

$$\alpha^4 |\text{Spec}_\alpha(A)| \mathbf{P}_Z(A) \leq \frac{E(A, A)}{|A|^3} \leq |\text{Spec}_\alpha(A)| \mathbf{P}_Z(A) + \alpha^2.$$

Thus, large energy forces large spectrum (and conversely).

- 4.6.4 Let $0 < \alpha \leq 1$, and let A, B be additive sets in Z with $|A| = |B| = N$ and $E(A, B) \geq 4\alpha^2 N^3$. Show that $|\text{Spec}_\alpha(A) \cap \text{Spec}_\alpha(B)| \geq \frac{2\alpha^2 |Z|}{N}$. Thus pairs of sets with large additive energy must necessarily have a large amount of shared spectrum.
- 4.6.5 If A is an additive set in a finite additive group Z , and A' is an additive set in a finite additive group Z' , show that $\text{Spec}_\alpha(A) \times \text{Spec}_\beta(A') \subseteq \text{Spec}_{\alpha\beta}(A \times A')$ for all $0 < \alpha, \beta \leq 1$, where we give $Z \times Z'$ the bilinear form induced from Z and Z' .