

- 4.6.6 Show that Theorem 4.41 implies Corollary 2.62. (Hint: use (4.14).)
- 4.6.7 Let S be a subset of a finite additive group Z , and let $0 < \rho < 1/4$. Show that if A is any additive set in $\text{Bohr}(S, \rho)$, then $S \subseteq \text{Spec}_{\cos(\pi\rho)}(A)$. This can be viewed as a kind of converse to Proposition 4.39.

4.7 Progressions in sum sets

A cornerstone of additive combinatorics is Szemerédi's theorem. One form of this theorem states that if A is a subset of the interval $[1, N]$ with positive density α , then A contains an arithmetic progression of length $f(N, \alpha)$, where f tends to infinity as N does and α is fixed. In Chapters 10 and 11, we will discuss this result in more detail, but let us mention here that f tends to infinity very slowly as a function of N .

In this section, we are going to show that if we replace the additive set A by a larger set, such as $A + B$, $A + A + A$, or $2A - 2A$, then one can locate significantly larger progressions inside these sets by taking advantage of the existence of functions supported on those sets with good Fourier transform, namely $1_A * 1_B$, $1_A * 1_A * 1_A$ and $1_A * 1_A * 1_{-A} * 1_{-A}$.

To illustrate this, we begin with a theorem of Chang (based on earlier work of Ruzsa [295]) which demonstrates the existence of a large generalized progression inside $2A - 2A$; this theorem will be a key ingredient in one of the formulations of Freiman's theorem (see Theorem 5.30).

Theorem 4.42 (Chang's theorem) [48] *Let $K, N \geq 1$. Let A be an additive set in a cyclic group $Z = \mathbf{Z}_N$ such that $E(A, A) \geq |A|^3/K$. Then there exists a proper progression $P \subseteq 2A - 2A$ of rank at most $O(K(1 + \log \frac{1}{\mathbf{P}_Z(A)}))$ and size*

$$|P| \geq O\left(K\left(1 + \log \frac{1}{\mathbf{P}_Z(A)}\right)\right)^{-O(K(1 + \log \frac{1}{\mathbf{P}_Z(A)}))} N. \quad (4.41)$$

Furthermore we may choose P to be symmetric ($-P = P$).

Note from (2.8) that the hypothesis $E(A, A) \geq |A|^3/K$ will be obeyed if $|A + A| \leq K|A|$ or $|A - A| \leq K|A|$; thus this theorem covers the case of sets with small doubling constant or small Ruzsa diameter. Alternatively, from the trivial bound $E(A, A) \geq |A|^2$ we see this hypothesis is always satisfied with $K = 1/\mathbf{P}_Z(A)$, but this is costly as the dependence of (4.41) on K is exponential. On the other hand, if A has small doubling then this theorem can be applied efficiently even when A is a rather sparse subset of Z .

Proof Set $\alpha := 1/2K^{1/2}$. By Proposition 4.39, we have

$$\text{Bohr} \left(\text{Spec}_\alpha(A), \frac{1}{2} \right) \subseteq 2A - 2A.$$

On the other hand, from Lemma 4.36 we can find a set $S := \{\eta_1, \dots, \eta_d\}$ of frequencies with

$$d = |S| = O \left(\alpha^{-2} \left(1 + \log \frac{1}{\mathbf{P}_Z(A)} \right) \right) = O \left(K \left(1 + \log \frac{1}{\mathbf{P}_Z(A)} \right) \right)$$

such that

$$\text{Spec}_\alpha(A) \subseteq [-1, 1]^d \cdot (\eta_1, \dots, \eta_d).$$

This implies (from the triangle inequality) that

$$\text{Bohr} \left(S, \frac{1}{6d} \right) \subseteq \text{Bohr} \left(\text{Spec}_\alpha(A), \frac{1}{6} \right).$$

Applying Proposition 4.23 we see that $\text{Bohr}(S, \frac{1}{6d})$ contains a proper symmetric progression of rank d and cardinality

$$|P| \geq \frac{(1/6d)^d}{d^d} N \geq O \left(K \left(1 + \log \frac{1}{\mathbf{P}_Z(A)} \right) \right)^{-O(K(1+\log \frac{1}{\mathbf{P}_Z(A)}))} N$$

and the claim follows. \square

In the proof of the above theorem (or more precisely, in the proof of Proposition 4.39) one took advantage of the fact that $1_A * 1_A * 1_{-A} * 1_{-A}$ had positive Fourier coefficients $|\widehat{1_A}(\xi)|^4$. However, it turns out that with a slight modification to the argument one does not need positivity of the Fourier coefficients, and in fact one only needs three summands instead of four:

Theorem 4.43 [149] *Let $K, N \geq 1$. Let A_1, A_2, A_3 be additive sets in \mathbf{Z}_N such that $|A_1| = |A_2| = |A_3|$ and $|A_1 + A_2 + A_3| \leq K|A_1|$. Then there exists a proper progression $P \subseteq A_1 + A_2 + A_3$ of rank at most $O(K^2(1 + \log \frac{1}{\mathbf{P}_Z(A_1)}))$ and size*

$$|P| \geq O \left(K \left(1 + \log \frac{1}{\mathbf{P}_Z(A_1)} \right) \right)^{-O(K^2(1+\log \frac{1}{\mathbf{P}_Z(A_1)}))} N. \quad (4.42)$$

One can of course generalize the hypotheses to deal with sets A_1, A_2, A_3 of differing cardinalities, but the statement of the theorem becomes a little messier and we do not pursue it here.

Proof We adapt some arguments of [117]. We consider the non-negative function $f := 1_{A_1} * 1_{A_2} * 1_{A_3}$. From (4.10) we have $\mathbf{E}_Z f = \mathbf{P}_Z(A_1)^3$. On the other hand, we have $\mathbf{P}_Z(\text{supp}(f)) = \mathbf{P}_Z(A_1 + A_2 + A_3) = K\mathbf{P}_Z(A_1)$. By the pigeonhole principle, we can thus find an element $x_0 \in A_1 + A_2 + A_3$ such that $f(x_0) \geq \mathbf{P}_Z(A_1)^2/K$. By translating one of the A_j , if necessary, we may assume $x_0 = 0$, thus $f(0) \geq \mathbf{P}_Z(A_1)^2/K$.

Next, we observe from (4.9) that $\hat{f}(\xi) = \widehat{1_{A_1}}(\xi)\widehat{1_{A_2}}(\xi)\widehat{1_{A_3}}(\xi)$. From (4.4), Cauchy–Schwarz, (4.16) and (4.24) we thus have for any $x \in Z$

$$\begin{aligned} |f(x) - f(0)| &= \left| \sum_{\xi \in Z} \widehat{1_{A_1}}(\xi)\widehat{1_{A_2}}(\xi)\widehat{1_{A_3}}(\xi)(e(\xi \cdot x) - 1) \right| \\ &\leq \sum_{\xi \in Z} |\widehat{1_{A_1}}(\xi)| |\widehat{1_{A_2}}(\xi)| |\widehat{1_{A_3}}(\xi)| |e(\xi \cdot x) - 1| \\ &\leq \left(\sup_{\xi \in Z} |\widehat{1_{A_1}}(\xi)| |e(\xi \cdot x) - 1| \right) \|\widehat{1_{A_2}}\|_{L^2(Z)} \|\widehat{1_{A_3}}\|_{L^2(Z)} \\ &= \mathbf{P}_Z(A_1) \sup_{\xi \in Z} |\widehat{1_{A_1}}(\xi)| |e(\xi \cdot x) - 1| \\ &\leq 2\pi \mathbf{P}_Z(A_1) \sup_{\xi \in Z} |\widehat{1_{A_1}}(\xi)| \|\xi \cdot x\|_{\mathbf{R}/\mathbf{Z}}. \end{aligned}$$

Combining this with our bound on $f(0)$ and the support of f , we see that

$$\left\{ x \in Z : \sup_{\xi \in Z} |\widehat{1_{A_1}}(\xi)| \|\xi \cdot x\|_{\mathbf{R}/\mathbf{Z}} < \mathbf{P}_Z(A_1)/2\pi K \right\} \subseteq A_1 + A_2 + A_3.$$

Since $|\widehat{1_{A_1}}(\xi)| \|\xi \cdot x\|_{\mathbf{R}/\mathbf{Z}} < \mathbf{P}_Z(A_1)/2\pi K$ whenever $\xi \notin \text{Spec}_{1/2\pi K}(A_1)$, we obtain

$$\left\{ x \in Z : \sup_{\xi \in \text{Spec}_{1/2\pi K}(A_1)} |\widehat{1_{A_1}}(\xi)| \|\xi \cdot x\|_{\mathbf{R}/\mathbf{Z}} < \mathbf{P}_Z(A_1)/2\pi K \right\} \subseteq A_1 + A_2 + A_3.$$

Moreover, as $|\widehat{1_{A_1}}(\xi)| \leq \mathbf{P}_Z(A_1)$ for all non-zero ξ , we obtain

$$\text{Bohr}(\text{Spec}_{1/2\pi K}(A_1), 1/2\pi K) \subseteq A_1 + A_2 + A_3$$

(for instance). But by Lemma 4.36 we can find $d = O(K^2(1 + \log \frac{1}{\mathbf{P}_Z(A_1)}))$ and frequencies $S := \{\eta_1, \dots, \eta_d\} \subset Z$ such that

$$\text{Spec}_{1/2\pi K}(A_1) \subseteq [-1, 1]^d \cdot (\eta_1, \dots, \eta_d)$$

and hence by the triangle inequality

$$\text{Bohr}(S, 1/2\pi dK) \subseteq \text{Bohr}(\text{Spec}_{1/2\pi K}(A_1), 1/2\pi K) \subseteq A_1 + A_2 + A_3.$$

Applying Proposition 4.23, we can locate a proper progression P in $\text{Bohr}(S, 1/2\pi dK)$ of rank d and cardinality at least

$$|P| \geq \frac{(1/2dK)^d}{d^d} N \geq (CK(1 + \log(1/\mathbf{P}_Z(A_1))))^{-CK^2(1+\log(1/\mathbf{P}_Z(A_1)))} N$$

and the claim follows. \square

The above arguments relied crucially on having three or more summands; roughly speaking, two of the summands were treated by Plancherel’s theorem, leaving at least one other summand to be free to exploit the smallness of its Fourier coefficients outside of its spectrum. They break down quite significantly for sums of two sets¹. Nevertheless, it is still possible to obtain some relatively large progressions in a set of the form $A + B$, because the function $1_A * 1_B$ still has l^1 type control on the Fourier coefficients. We follow the arguments of Bourgain [36]. We first give a convenient criterion for establishing the existence of progressions.

Lemma 4.44 (Almost periodicity implies long progressions)[36] *Let $f : Z \rightarrow \mathbf{R}^+$ be a non-negative random variable on an additive group Z , let $J \geq 1$ be an integer, and suppose that $r \in Z$ is such that*

$$\mathbf{E}_Z \max_{1 \leq j \leq J} |T^{jr} f - f| < \mathbf{E}_Z f,$$

where $T^{jr} f(x) := f(x - jr)$ is the shift of f by jr . Then $\text{supp}(f)$ contains an arithmetic progression $a + [0, J] \cdot r$ of length $J + 1$ and spacing r .

Proof By the pigeonhole principle, there exists $x \in Z$ such that

$$\max_{1 \leq j \leq J} |T^{jr} f(x) - f(x)| < f(x)$$

and hence $f(x - jr) = T^{jr} f(x) > 0$ for all $0 \leq j \leq J$. The claim follows. \square

To apply this lemma, we need to estimate expressions of the form $\mathbf{E}_Z \max_{1 \leq j \leq J} |T^{jr} f - f|$. This can be done easily if f has Fourier transform in a dissociated set:

Lemma 4.45 [36] *Let $S \subseteq Z$ be a dissociated set, and let f be a random variable such that $\text{supp}(\hat{f}) \subseteq S$. Then for any non-empty set of shifts $H \subset Z$ we have*

$$\left\| \max_{h \in H} |T^h f| \right\|_{L^2(Z)} = O(1 + \log |H|)^{1/2} \|f\|_{L^2(Z)}.$$

¹ There is a similarity with the Goldbach conjectures. The weak conjecture – every large odd number is the sum of three primes – has been solved by Fourier methods, whereas the strong conjecture – every large even number is the sum of two primes – is still open, and probably not amenable to a purely Fourier-analytic method.

Proof Let $p > 2$ be a large exponent to be chosen later. Then

$$\begin{aligned} \left\| \max_{h \in H} |T^h f| \right\|_{L^2(Z)} &\leq \left\| \max_{h \in H} |T^h f| \right\|_{L^p(Z)} \\ &\leq \left\| \left(\sum_{h \in H} |T^h f|^p \right)^{1/p} \right\|_{L^p(Z)} \\ &= \sum_{h \in H} \|T^h f\|_{L^p(Z)}^{1/p} \\ &\leq |H|^{1/p} \|f\|_{L^p(Z)} \\ &\leq |H|^{1/p} \|S\|_{\Lambda(p)} \|f\|_{L^2(Z)} \\ &= O(|H|^{1/p} \sqrt{p} \|f\|_{L^2(Z)}) \end{aligned}$$

by Rudin's inequality (Lemma 4.33). The claim now follows by setting $p := O(1 + \log |H|)$. \square

By combining this lemma with Lemma 4.35, we can obtain an estimate when $\text{supp}(\hat{f})$ is not dissociated, but \hat{f} is uniform in size:

Lemma 4.46 [36] *Let f be a random variable, and let $J, d > 1$. Suppose that there exists an integer m such that $2^m \leq |\hat{f}(\xi)| \leq 2^{m+1}$ for all $\xi \in \text{supp}(\hat{f})$. Then one can find a set $S \subset Z$ of cardinality $|S| = d$ such that such that*

$$\mathbf{E}_Z \max_{1 \leq j \in J} |T^{jr} f - f| = O \left(\sum_{\xi \in \text{supp}(\hat{f})} |\hat{f}(\xi)| \left(\sqrt{\frac{\log J}{d}} + Jd \max_{\eta \in S} \|\eta \cdot r\|_{\mathbf{R}/Z} \right) \right)$$

for all $r \in Z$.

Proof Applying Lemma 4.35, we may write

$$\text{supp}(\hat{f}) = D_1 \cup \dots \cup D_k \cup R$$

where D_1, \dots, D_k are disjoint dissociated sets of cardinality $d + 1$, and $R \subseteq [-1, 1]^d \cdot (\eta_1, \dots, \eta_d)$ for some $S = \{\eta_1, \dots, \eta_d\} \subset Z$. Using the Fourier transform, we may then split $f = f_{D_1} + \dots + f_{D_k} + f_R$ accordingly. From Lemma 4.45 we have, for any $1 \leq i \leq k$,

$$\begin{aligned} \mathbf{E}_Z \max_{1 \leq j \in J} |T^{jr} f_{D_i} - f_{D_i}| &\leq 2 \left\| \max_{0 \leq j \in J} |T^{jr} f_{D_i}| \right\|_{L^2(Z)} \\ &\leq O(\log^{1/2} J \|f_{D_i}\|_{L^2(Z)}) \\ &= O \left(\log^{1/2} J \left(\sum_{\xi \in D_i} |\hat{f}(\xi)|^2 \right)^{1/2} \right) \\ &\leq O \left(\sqrt{\frac{\log J}{D}} \sum_{\xi \in D_i} |\hat{f}(\xi)| \right) \end{aligned}$$

thanks to the uniformity assumption $2^m \leq |\hat{f}(\xi)| \leq 2^{m+1}$. Also, we have from the triangle inequality, (4.24) and the hypothesis on R

$$\begin{aligned} \left\| \max_{1 \leq j \leq J} |T^{jr} f_R - f_R| \right\|_{L^1(Z)} &\leq \left\| \max_{1 \leq j \leq J} \sum_{\xi \in R} |\hat{f}(\xi)| \times |e(x + jr, \xi) - e(\xi \cdot x)| \right\|_{L^1(Z)} \\ &\leq \left(\sum_{\xi \in R} |\hat{f}(\xi)| \right) \max_{1 \leq j \leq J; \xi \in R} |e(jr, \xi) - 1| \\ &\leq 2\pi Jd \left(\sum_{\xi \in R} |\hat{f}(\xi)| \right) \max_{\eta \in S} \|\eta \cdot r\|_{\mathbf{R}/Z}. \end{aligned}$$

Summing these estimates using the triangle inequality, the claim follows. □

Now we can prove Bourgain’s theorem.

Theorem 4.47 [36] *Let $N \geq 1$ be a prime number, and let A, B be additive sets in \mathbf{Z}_N such that $|A|, |B| \geq \delta N$ for some $C \frac{(\log \log N)^3}{\log N} < \delta \leq 1$ for some large absolute constant $C > 1$. Then $A + B$ contains a proper arithmetic progression of length at least $\exp(\Omega(\delta \log N)^{1/3})$.*

Proof We may assume N to be large. By removing elements from A and B and increasing δ if necessary we may assume $\mathbf{P}_Z(A) = \mathbf{P}_Z(B) = \delta$. Set $f := 1_A * 1_B$, and let $\exp(\Omega(\delta \log N)^{1/3}) \leq J < N$ be chosen later: thus $\text{supp}(f) = A + B$ and $\mathbf{E}_Z f = \mathbf{P}_Z(A)\mathbf{P}_Z(B) = \delta^2$; note also that $J \gg 1/\delta$. By Lemma 4.44, it suffices to show that

$$\mathbf{E}_Z \max_{1 \leq j \leq J} |T^{jr} f - f| < \delta^2$$

for some non-zero r .

The Fourier coefficients \hat{f} of f cannot exceed $\hat{f}(0) = \mathbf{E}_Z f = \delta$. Furthermore we have by Cauchy–Schwarz and (4.16)

$$\begin{aligned} \sum_{\xi \in Z} |\hat{f}(\xi)| &= \sum_{\xi \in Z} |\widehat{1_A}(\xi)| |\widehat{1_B}(\xi)| \\ &\leq \|\widehat{1_A}\|_{l^2(Z)} \|\widehat{1_B}\|_{l^2(Z)} \\ &= \mathbf{P}_Z(A)^{1/2} \mathbf{P}_Z(B)^{1/2} \\ &= \delta. \end{aligned} \tag{4.43}$$

To exploit this, we let $M \geq 1$ be chosen later and partition

$$Z = \bigcup_{0 \leq m < M} \Gamma_m \cup \Gamma_{err}$$

where $\Gamma_m := \{\xi \in Z : 2^{-m-1}\delta^2 < |\hat{f}(\xi)| \leq 2^{-m}\delta^2\}$ and $S_{err} := \{\xi \in Z : |\hat{f}(\xi)| \leq 2^{-M}\delta^2\}$. This induces a splitting

$$f = \sum_{0 \leq m \leq 10 \log \frac{1}{\delta}} f_m + f_{err}.$$

We can apply Lemma 4.46 to each f_m , with $d \geq 1$ to be chosen later, to obtain

$$\mathbf{E}_Z \max_{1 \leq j \in J} |T^{jr} f_m - f_m| = O \left(\sum_{\xi \in \Gamma_m} |\hat{f}_m(\xi)| \left(\sqrt{\frac{\log J}{d}} + Jd \max_{\eta \in S_m} \|\eta \cdot r\|_{\mathbf{R}/\mathbf{Z}} \right) \right)$$

where S_m is a set of frequencies of cardinality $|S_m| = d$; summing this in m and using (4.43) we obtain

$$\sum_{0 \leq m < M} \mathbf{E}_Z \max_{1 \leq j \in J} |T^{jr} f_m - f_m| = O \left(\delta \left(\sqrt{\frac{\log J}{d}} + Jd \max_{\eta \in S} \|\eta \cdot r\|_{\mathbf{R}/\mathbf{Z}} \right) \right)$$

where $S := \bigcup_{0 \leq m < M} S_m$ is a set of frequencies of cardinality $|S| \leq dM$. As for f_{err} , we crudely use the triangle inequality:

$$\begin{aligned} \mathbf{E}_Z \max_{1 \leq j \in J} |T^{jr} f_{err} - f_{err}| &\leq \sum_{1 \leq j \leq J} \|T^{jr} f_{err}\|_{L^1(Z)} \\ &\leq \sum_{1 \leq j \leq J} \|T^{jr} f_{err}\|_{L^2(Z)} \\ &= J \left(\sum_{\xi \in \Gamma_{err}} |\hat{f}(\xi)|^2 \right)^{1/2} \\ &\leq J 2^{-M} \delta \|\hat{f}\|_{l^2(Z)} \\ &\leq J 2^{-M/2} \delta. \end{aligned}$$

Combining these estimates using the triangle inequality, we see that to conclude the theorem we need to find an $r \neq 0$ such that

$$\sqrt{\frac{\log J}{d}} + Jd \max_{\eta \in S} \|\eta \cdot r\|_{\mathbf{R}/\mathbf{Z}} + J 2^{-M/2} < c\delta$$

for some small absolute constant $c > 0$. If we choose $M := C \log J$ and $d := C\delta^{-2} \log J$ for a sufficiently large C , then it is clear the first and third terms will be less than $c\delta/3$ (recalling that $J \gg 1/\delta$), and so it will suffice to find an $r \neq 0$ such that

$$\max_{\eta \in S} \|\eta \cdot r\|_{\mathbf{R}/\mathbf{Z}} < \frac{c\delta}{3Jd} < \frac{c'\delta^3}{J \log J}$$

where $c' > 0$ is another small absolute constant. Using Lemma 4.20, we see that this is possible provided that

$$2^{-|S|} \left(\frac{c' \delta^3}{J \log J} \right)^{|S|} N > 1.$$

But since

$$|S| \leq dM = O(\delta^{-2} \log^2 J)$$

we see that we can achieve this by setting $J := \exp(c''(\delta \log N)^{1/3})$ for a suitably small c'' , using the lower bound hypothesis on δ . The claim follows. \square

The length $\exp(\Omega(\delta \log N)^{1/3})$ was recently extended to $\exp(\Omega(\delta \log N)^{1/2})$ (and the condition on δ relaxed slightly to $C \frac{(\log \log N)^2}{\log N} < \delta$ by Green [149], by an interesting variational Fourier argument which we briefly sketch here. The starting point is Exercise 4.3.12, non-empty set E of some fixed density $\mathbf{P}_Z(E) = \beta$, to be chosen later, which is disjoint from $A + B$ and minimizes the quantity $\|E\|_u$ subject to the above constraints; Exercise 4.3.12 thus places a lower bound on this quantity $\|E\|_u$. One then considers the α -spectrum $\Lambda_\alpha(E)$ of E , for some α to be chosen later, and uses Lemma 4.36 to place this spectrum inside a progression $[-1, 1]^d \cdot (\eta_1, \dots, \eta_d)$ for some set $S = \{\eta_1, \dots, \eta_d\}$ of frequencies which is not too large. Next, one removes a small number of elements (chosen at random) from E and replace them by generic elements of Z ; by Lemma 4.16 this shrinks the Fourier bias of E with high probability. Next, one takes these new generic elements of Z and translates them by a suitable element of $\text{Bohr}(S, \rho)$ (for some suitably small ρ) to try to place all of them outside $A + B$. This operation, if successful, will not significantly affect the Fourier transform of E on the large spectrum $\Lambda_\alpha(E)$ and should thus still shrink the Fourier bias of E . But this contradicts the construction of E . Thus it must not be possible to translate one of the generic elements outside of $A + B$, which means that $A + B$ necessarily contains a translate of $\text{Bohr}(S, \rho)$. From this and Proposition 4.23 one then establishes a large progression inside $A + B$. For more details (such as the selection of the parameters β, α, ρ), see [149].

On the other hand, an example of Ruzsa [290] shows that even when δ is close to $1/2$, one can find sets $A + A$ which do not contain any progressions of length $\exp(\Omega(\log N)^{2/3})$.

The arithmetic progressions inside iterated sum sets have been intensively studied in [350]; we discuss this in detail in Chapter 12.

Exercises

4.7.1 [149] Let $A_1, A_2, A_3 \subset [1, N]$ be additive sets of integers such that $|A_1| = |A_2| = |A_3| \geq \delta N$ for some $0 < \delta < 1/2$. Show that $A_1 + A_1$

contains a proper arithmetic progression of length at least $\exp(\Omega(\delta \log N)^{1/3} - O(\log \log N))$, $A_1 + A_2 + A_3$ contains a proper arithmetic progression of length at least $O(\delta^{O(1)} N^{\delta^2/\log 1/\delta})$ and that $2A_1 - 2A_1$ contains a proper arithmetic progression of length at least $O(\delta^{O(1)} N^{\delta/\log 1/\delta})$. (Hint: embed A_1, A_2, A_3 in \mathbf{Z}_p for some prime $2N < p < 4N$ and apply the theorems of this section, followed by Exercise 3.2.5. One needs to pass from a progression in \mathbf{Z}_N back to one in \mathbf{Z} ; one tool for this is Corollary 3.25.)

- 4.7.2 [349] Let P be a proper arithmetic progression in a torsion-free additive group, and let A, B be additive sets in P such that $|A|, |B| > (1 - \varepsilon)|P|$ for some $0 < \varepsilon < 1/4$. Prove that $A + B$ contains a proper arithmetic progression of length at least $(2 - 4\varepsilon)|P| - 1$. (Hint: work with those elements of $P + P$ which have at least $2\varepsilon|P|$ representations as sums of elements of P .)