5.1.14 Show that Theorem 5.11 fails when $|2A| = 3|A| - 3$, by considering a progression of rank 2. Also show that the quantity $2|A| - |A|$ in that theorem cannot be replaced by any smaller quantity.

5.1.15 Let A, B be additive sets of integers. If $A \hat{+} B := \{a + b : a, b \in A, a \neq b\}$ denotes the restricted sum set of A and B, show that $|A \hat{+} B| \geq |A| + |B| - 3$. (Hint: a direct application of the e-transform will not work, but if one deconstructs the proof of Lemma 5.3 one can modify it to deal with restricted sum sets.) If $|A| \neq |B|$, improve the preceding bound to $|A \hat{+} B| \geq |A| + |B| - 2$. (Hint: one needs to adapt some ideas from Proposition 5.8.) An analogous result for $\mathbf{Z}_p$ is known, but requires more non-elementary methods; see Section 9.2.

5.2 Sum sets in vector spaces

We now study the minimal size of sum sets in a real finite-dimensional vector space V, exploiting such concepts as convexity which are not readily available in other groups. Of course, since V contains a copy of $\mathbf{Z}$, we know from Lemma 5.3 that $|A + B|$ can be as small as $|A| + |B| - 1$. However, one can do better than this if one knows that $A + B$ is high-dimensional, or in other words that it is not contained in a low-dimensional affine vector space (a translate of a linear vector space).

We begin with the case $A = B$, which is somewhat easier. Define the *rank* $\mathrm{rank}(A)$ of a subset of V to be the smallest d such that A is contained in an affine space of dimension d.

Lemma 5.13 (Frieman's lemma) *[116] Let A be an additive set in a finite-dimensional vector space V, and let suppose that* $\mathrm{rank}(A) \geq d$ *for some $d \geq 1$. Then we have*

$$|A + A| \geq (d + 1)|A| - \frac{d(d + 1)}{2}.$$

Proof We induce on d. If $d = 1$ then the claim follows from Theorem 5.5, so let us assume $d \geq 2$ and that the claim is already proven for $d - 1$. Now we fix d and induce on $|A|$. The claim is vacuously true if say $|A| = 1$, so assume $|A| \geq 2$ and that the claim is already been proven for smaller sets A. Let $a \in A$ be any extreme point of A; thus a is a vertex on the convex hull of A. Let $A' := A - \{a\}$. We divide into two cases. If $\mathrm{rank}(A') \geq d$, then by induction hypothesis

$$|A' + A'| \geq (d + 1)|A'| - \frac{d(d + 1)}{2}.$$

Since a lies outside of the convex hull of A' and $\mathrm{rank}(A') \geq d$, there must exist (by the greedy algorithm) at least d extreme points $x_1, \ldots, x_d$ of A' which are visible from a in the sense that the line segments joining a to $x_1, \ldots, x_d$ lie outside the convex hull of A'. In particular we see that the $d+1$ points $a, \frac{a+x_1}{2}, \ldots, \frac{a+x_d}{2}$ lie outside the convex hull of A', and in particular outside of $\frac{1}{2} \cdot (A' + A')$. Dilating this by 2 we see that $a+a, a+x_1, \ldots, a+x_d$ are disjoint from $A' + A'$. Thus

$$|A + A| \geq d + 1 + |A' + A'| \geq (d+1)|A| - \frac{d(d+1)}{2}$$

thus closing the induction.

It remains to consider the case when $\mathrm{rank}(A') < d$, thus A is contained in a $d-1$-dimensional affine space W. Since $\mathrm{rank}(A) \geq d$, we have $a \notin W$. This means that $2a$, $a + W$, and $2W$ are all disjoint; thus $a + a$, $a + A'$, and $A' + A'$ are all disjoint; thus

$$|A + A| \geq 1 + |A| - 1 + |A' + A'|.$$

But since $\mathrm{rank}(A) \geq d$, we have $\mathrm{rank}(A') = \mathrm{rank}(A \backslash \{a\}) \geq d - 1$, and hence by induction

$$|A' + A'| \geq d|A'| - \frac{d(d-1)}{2} = d|A| - \frac{d(d+1)}{2}$$

and the claim again follows by induction. □

Now we consider the problem of sums of two sets A, B in V. To make this problem more precise, let us temporarily define the quantity $S(d, n, t)$ for any $n \geq 1$, $t \geq 0$, and $d \geq 0$, to be the least value of $|A + B|$, where A, B ranges over all additive sets in a finite-dimensional vector space V, such that $|A| \geq n$, $|B| \geq n - t$, and $\mathrm{rank}(A + B) \geq d$. Since $|A + B| \geq |A|$ we have the trivial bound

$$S(d, n, t) \geq n. \tag{5.12}$$

This bound is however not sharp in general, and we shall improve it presently. We first need a lemma analyzing the behavior of $A + B$ near an extreme point of A and B, similar to that used in the proof of Lemma 5.13.

Lemma 5.14 *[296] Let A, B be additive sets in a finite-dimensional vector space V such that A and B both contain* 0, *and suppose that* 0 *is a vertex on the convex hull of $A \cup B$. Let $A' := A - \{0\}$ and $B' := B - \{0\}$, and $C := (A' \cup B') \backslash (A' + B')$. Then $A + B$ lies in the subspace of V spanned by C.*

Proof Without loss of generality we may take $V = \mathbf{R}^n$. By the Hahn–Banach theorem, there exists a linear functional $\phi : V \to \mathbf{R}$ such that $\phi(x) > 0$ for all $x \in (A \cup B) \backslash 0$. We need to show that every element x of $A + B$ lies in the span of C. We shall prove this by induction on $\phi(x)$, which is a non-negative integer. If $\phi(x) = 0$, then $x = 0$ and there is nothing to prove. Now suppose that $\phi(x) > 0$ and the claim has already been shown for all smaller values of $\phi(x)$. If $x \in A' + B'$, then we can write $x = a + b$ where $a \in A'$ and $b \in B'$. But since $\phi(x) = \phi(a) + \phi(b)$ and $\phi(a), \phi(b) > 0$, we see that $\phi(a), \phi(b)$ are strictly less than $\phi(x)$, and the claim follows from induction. The only remaining case is when $\phi(x) > 0$ and $x \notin (A' + B')$. But since $x \in A + B$, this implies that $x \in C$, and we are done. □

We can now obtain the following recursive inequality on $S(d, n, t)$.

Proposition 5.15 *[296] Let $d \geq 1$, $n \geq 2$, and $t \leq n - 2$. Then*

$$S(d, n, t) \geq \min(S(d, n-1, t) + d + 1, S(d-1, n-1, t) + n, \\ S(d-1, n-1, t-1) + n - t).$$

Proof Let A, B be as in the definition of $S(d, n, t)$; note that A and B contain at least two elements. Since A and B are finite, we can find a linear functional $\phi : V \to \mathbf{R}$ which is injective on $A \cup B$ (indeed one could select ϕ randomly). Since ϕ is injective, we see that there is a unique element $a_0 \in A$ which minimizes ϕ on A, i.e. $\phi(a) > \phi(a_0)$ for all $a \in A \backslash a_0$. Similarly we can find a $b_0 \in B$ which minimizes ϕ on B, so that $\phi(b) > \phi(b_0)$ for all $b \in B \backslash b_0$. By translating A and B if necessary we may assume $a_0 = b_0 = 0$. Thus A and B now both contain 0, and if we define $A' := A \backslash \{0\}$ and $B' := B \backslash \{0\}$, then ϕ is strictly positive on both A' and B'. In particular ϕ is strictly positive on $A' + B'$, which therefore does not contain 0.

From Lemma 5.14 we have

$$|(A' \cup B') \backslash (A' + B')| \geq d$$

and hence (since 0 is contained in $A + B$ but not A', B', or $A' + B'$)

$$|A + B| \geq |A' + B'| + d + 1.$$

Let $c + W$ denote the affine span of $A' + B'$, where $c \in V$ and W is a linear subspace of W. If we knew that $\mathrm{rank}(A' + B') = \dim(W) \geq d$, we could then conclude that $|A' + B'| \geq S(d, n-1, t)$, and we would be done. Thus we may assume that $\dim(W) \leq d - 1$. Thus if we pick $a_1 \in A'$ and $b_1 \in B'$ arbitrarily, then we have $A' \in a_1 + W$ and $B' \in b_1 + W$. Thus $A + B$ is contained in the span of W, a_1, and b_1. By hypothesis, this means that at least one of a_1, b_1 must lie outside of W.

We now divide into a number of cases depending of the relative position of a_1 and b_1 with respect to W. Suppose first that a_1 and b_1 are linearly independent modulo W. Then $A = 0 \cup A'$ lies in $\{0, a_1\} + W$, and is thus disjoint from $A + B'$, which lies in $\{b_1, a_1 + b_1\} + W$; so

$$|A + B| \geq |A + B'| + |A| \geq |A + B'| + n.$$

On the other hand, $\mathrm{rank}(A + B') \geq \mathrm{rank}(A + B) - 1 \geq d - 1$, which implies $|A + B'| \geq S(d - 1, n - 1, t)$. The claim thus follows in this case.

Now suppose that a_1, b_1 are linearly dependent modulo W and $b_1 \notin W$. Then $A' \subset a_1 + W$ and $A' + B' \subset a_1 + b_1 + W$ are disjoint, while 0 is disjoint from A' (by definition) and $A' + B'$ (by previous remarks). Thus

$$|A + B| \geq 1 + |A'| + |A' + B'| \geq n + |A' + B'|.$$

On the other hand, since $A + B$ is contained in the span of W and b_1, we have $\mathrm{rank}(A' + B') = \dim(W) \geq \mathrm{rank}(A + B) - 1 \geq d - 1$, hence $|A' + B'| \geq S(d - 1, n - 1, t)$. The claim again follows.

The only remaining case is when $b_1 \in W$, which forces $a_1 \notin W$ by previous discussion. Then $A' + B$ and B are disjoint, thus

$$|A + B| \geq |B| + |A' + B| \geq n - t + |A' + B|.$$

But since $\mathrm{rank}(A' + B) \geq \mathrm{rank}(A + B) - 1 \geq d - 1$, we have $|A' + B| \geq S(d - 1, n - 1, t - 1)$, and the claim again follows. □

Corollary 5.16 *[296] For any $n \geq 1$, $t \geq 0$, $d \geq 0$ we have*

$$S(d, n, t) \geq \sum_{n-d \leq r \leq n} r - \sum_{1 \leq s \leq t} \min(s, d).$$

Proof The cases $d = 0$, $n = 1$, or $r \geq n - 1$ can be easily verified from (5.12), so we may restrict ourselves to the case $d \geq 1$, $n \geq 2$, and $t \leq n - 2$. We shall induce on the positive quantity $n + d + t$, assuming inductively that the claim has already been proven for all smaller values of $n + d + t$. But then we have

$$\begin{aligned}
S(d, n-1, t) + d + 1 &\geq \sum_{n-d-1 \leq r \leq n-1} r - \sum_{1 \leq s \leq t} \min(s, d) + d + 1 \\
&= \sum_{n-d \leq r \leq n} r - \sum_{1 \leq s \leq t} \min(s, d) \\
S(d-1, n-1, t) + n &\geq \sum_{n-d \leq r \leq n-1} r + n - \sum_{1 \leq s \leq t} \min(s, d-1) \\
&\geq \sum_{n-d \leq r \leq n} r - \sum_{1 \leq s \leq t} \min(s, d) \\
S(d-1, n-1, t-1) + n - t &\geq \sum_{n-d \leq r \leq n-1} r - \sum_{1 \leq s \leq t-1} \min(s, d) + n - t \\
&\geq \sum_{n-d \leq r \leq n} r - \sum_{1 \leq s \leq t} \min(s, d)
\end{aligned}$$

and the claim follows from Proposition 5.15. □

This inequality is sharp in many cases, although there have been some refinements using techniques relating to the Brunn–Minkowski inequality (Theorem 3.16); see [128], [129]. As a consequence of the inequality we obtain the following generalization of Theorem 5.13:

Theorem 5.17 *[296] Let V a finite-dimensional vector space and $d \geq 0$, and let A, B be additive sets in V such that* $\mathrm{rank}(A+B) \geq d$*, then have* $|A+B| \geq |A| + d|B| - \frac{d(d+1)}{2}$.

Proof Apply Corollary 5.16 with $n := |A|$, $t := |A| - |B|$ and use the trivial bound $\sum_{1 \leq s \leq t} \min(s, d) \geq t$ to obtain

$$|A+B| \geq (d+1)\left(n - \frac{d}{2}\right) - t = n + d(n-t) - \frac{d(d+1)}{2}$$

as desired. □

We now return to additive sets in a vector space with small doubling. Define a *d-parallelepiped P* in a vector space V to be any set of the form

$$P = a + \overline{I} \cdot v_1 + \cdots + \overline{I} \cdot v_d$$

where $v_1, \ldots, v_d$ are vectors in V (not necessarily linearly independent), $a \in V$, and $\overline{I} = \{x \in \mathbf{R} : -1 \leq x \leq 1\}$ is the closed unit interval. The 2^d points $a + \{-1, 1\} \cdot v_1 + \cdots + \{-1, 1\} \cdot v_d$ (which may possibly have multiplicity) are called the *corners* of this d-parallelepiped, while a is the *center*; note that the corners form a progression of rank d and dimensions $(2, \ldots, 2)$, which may or may not be proper. A remarkable fact, known as the *Freiman cube lemma*, is that if an additive set A in a d-dimensional vector space has small doubling, then there is a d-parallelepiped which contains a large fraction of A and whose corners lie in the set A. This is certainly not true for general sets A, as can be seen for instance by considering the set $\{(n, n^2) : -N \leq n \leq N\}$ in $\mathbf{Z}^2 \subset \mathbf{R}^2$. To prove the Freiman cube lemma we first prove an auxiliary lemma which is useful for inductive purposes:

Lemma 5.18 *[28] Let V be an d-dimensional vector space, and let W be a $d-r$-dimensional linear subspace of V for some $0 < r \leq d$. Let A be a symmetric additive set in V (thus $-A = A$) and let $K = \sigma[A] = |A+A|/|A|$ be the doubling constant. Then there exists a r-paralleopiped P with corners in A and center* 0 *such that*

$$|A \cap (P + W)| \geq (9K)^{-2^{r-1}+1}|A|.$$

Proof We induce on the codimension r. First suppose that $r = 1$. Without loss of generality we may take V to be a Euclidean space $\mathbf{R}^d$. We let v_1 be an element of A which maximizes the quantity $\mathrm{dist}(v_1, W)$; then it is easily seen that the 1-parallelepiped $P = 0 + \overline{I} \cdot v_1$ will obey the desired properties (here we exploit the symmetry of A to place both corners of P in A).

Now suppose that $r \geq 2$ and that the claim has already been proven for all smaller values of r. We place W inside a $d-1$-dimensional hyperplane $H \subset V$, which divides V into the hyperplane H and into two open half-spaces H_- and H_+. By the pigeonhole principle, one of the three sets $A \cap H$, $A \cap H_-$, and $A \cap H_+$ has cardinality at least $|A|/3$.

Suppose first that $|A \cap H| \geq |A|/3$. Then by applying the induction hypothesis (with V replaced by H and d replaced by $d-1$) we can find an $r-1$-parallelepiped $P \subset H \subset V$ with corners in $A \cap H \subseteq H$ and center 0 such that

$$\begin{aligned} |A \cap (P + W)| &\geq |(A \cap H) \cap (P + W)| \geq (9K)^{-2^{r-2}+1}|A|/3 \\ &\geq (9K)^{-2^{r-1}+1}|A|. \end{aligned}$$

The claim then follows by adding a dummy vector $v_r = 0$ to P to make it a r-parallelepiped.

Without loss of generality, it remains to consider the case when $|A \cap H_+| \geq |A|/3$. Since $|2(A \cap H_+)| \leq |2A| \leq K|A|$, we conclude that $\sigma[A \cap H_+] \leq 3K$. By Exercise 2.3.14, some origin $a = x/2$ (since $F = x - F$) with $|F| \geq |A|/9K$ and $\sigma[F] \leq 9K^2$. Since F is contained entirely in the half-space H_+, we see that $a \in H_+$ also. In particular, $a \notin W$. Now let W' be the $d - r + 1$-dimensional linear space spanned by W and a, and apply the induction hypothesis with A replaced by $F - a$, K replaced by $9K^2$, W replaced by W' and r replaced by $r-1$. This allows us to find a $r-1$-parallelepiped $P' = a + \overline{I} \cdot v_1 + \cdots + \overline{I} \cdot v_{r-1}$ with center a and corners in F such that

$$|F \cap (P' + W')| \geq (81K^2)^{-2^{-r-2}+1}|F| \geq (9K)^{-2^{-r-1}+1}|A|.$$

Now we let P be the r-parallelepiped $\overline{I} \cdot a + \overline{I} \cdot v_1 + \cdots + \overline{I} \cdot v_{r-1}$; since F and $-F$ are both contained in A (by the symmetry of A) we see that the corners of P lie in A, and P is certainly centered at the origin. To conclude the proof we need to show that

$$|A \cap (P + W)| \geq |F \cap (P' + W')|.$$

To prove this, we use a sliding argument taking advantage of the symmetries of A and F. Let us split $W' = W_{>0} \cup W_{\leq 0}$, where $W_{>0}$ is the open half-space in W' with boundary W which contains a, and $W_{\leq 0}$ is the closed half-space in W' with boundary W which excludes a. Then

$$\begin{aligned}|F \cap (P' + W')| &= |F \cap (P' + W_{>0})| + |F \cap (P' + W_{\leq 0})| \\ &= |(F - 2a) \cap (P' + W_{>0} - 2a)| + |F \cap (P' + W_{\leq 0})| \\ &= |(-F) \cap (P' + W_{>0} - 2a)| + |F \cap (P' + W_{\leq 0})| \\ &= |[(-F) \cap (P' + W_{>0} - 2a)] \cup [F \cap (P' + W_{\leq 0})]|\end{aligned}$$

since F is symmetric around a, and F and $-F$ are disjoint (one lies in H_+ and the other lies in H_-). It thus suffices to show that the sets $-F \cap (P' + W_{>0} - 2a)$ and $F \cap (P' + W_{\leq 0})$ lie in $A \cap (P + W)$. That these sets lie in A is clear, since A contains both F and $-F$. Also observe that $(P' + W_{>0} - 2a)$ is contained in $-(P' + W_{\leq 0})$ since P' is symmetric around a'. Thus it only remains to show that $F \cap (P' + W_{\leq 0}) \subseteq P + W$. But since $F = 2a - F$ lies in H_+, and the corners of P' lie in F, and W lies in H, we see that both F and $P' + W$ lie in the slab between H and $2a + H$. Thus $F \cap (P' + W_{\leq 0})$ lies in the set $P' - \{ta : 0 \leq t \leq 2\} + W = P + W$, and the claim follows. □

As a corollary we obtain

Corollary 5.19 (Freiman cube lemma) *Let A be an additive set in a d-dimensional vector space V, and let $K = \sigma[A]$ be the doubling constant. Then there exists a d-parallelepiped with corners in A such that $|A \cap P| \geq (3K)^{-2^d}|A|$.*

Proof [28] Applying Exercise 2.3.14 we have $\sigma[F] \leq K^2$. Now apply Lemma 5.18 with $W = \{0\}$ and $r = d$. □

Lemma 5.13 shows, roughly speaking, that if A is an additive set in a vector space then rank(A) is controlled by a linear function of the doubling constant $\sigma[A]$. The following remarkable theorem shows that if one is willing to pass from A to a significant subset of A, then one can in fact control the rank by a *logarithmic* function of the doubling constant.

Theorem 5.20 (Freiman 2^n theorem) *Let $d \geq 1$, and let A be an additive set in a vector space V with doubling constant $K = \sigma[A] < 2^d$. Then there exists a subset A' of A with* rank$(A') < d$ *such that $\sigma[A'] \leq K$ and $|A'| = \Theta_{d,K}(|A|)$.*

See [28] for further discussion, including the dependence of constants in the $\Theta_{d,K}()$ notation.

Proof [28] We fix d and induce on K. For $K \leq 1$ the claim is vacuously true. Now suppose that $K > 1$ and that the claim has already been proven for values of $K \leq K - \varepsilon(d, K)$ for some $\varepsilon(d, K) > 0$ which is bounded from below for K in any compact interval $\{1 \leq K \leq 2^d - \delta\}$; if we can prove the claim under such a hypothesis, then the claim follows unconditionally by a standard continuity argument (the set of K obeying the theorem is open, closed, and contains 1).

Fix A, V, K, and let $\varepsilon = \varepsilon(d, K)$ be chosen later. If there exists a set $A'' \subset A$ with $|A''| \geq \varepsilon/K|A|$ and $\sigma[A''] \leq K - \varepsilon$, then the claim would follow by applying the induction hypothesis with A replaced by A'' and K by $K - \varepsilon$. Thus we may assume that $\sigma[A''] \geq K - \varepsilon$ whenever $|A''| \leq \varepsilon/K|A|$. In particular we see that

$$|2A''| \geq K|A''| - \varepsilon|A| \text{ for all non-empty } A'' \subseteq A \tag{5.13}$$

(treating the case of small A'' and large A'' separately). Note that this also holds with $A'' = \emptyset$ if we adopt the convention that $2A'' = \emptyset$ in this case.

Let $r = \mathrm{rank}(A)$. Without loss of generality we may assume that V is r-dimensional, since otherwise we can restrict V to the affine span of A (and translate to the origin). If A is small, say $|A| \leq 10K^2$, then the claim follows just by setting A' to be a single point, so assume $|A| > 10K^2$. By Lemma 5.13 we conclude $r \leq K$. We will in fact show that the hypotheses on A force $r \leq d$, at which point we can take $A' := A$ and be done.

We now claim that (5.13) implies the bound

$$|A \cap W| = O(\varepsilon|A|) \tag{5.14}$$

for all affine hyperplanes W in V. To see this, observe that W divides V into the hyperplane W and two open half-spaces W_-, W_+. Since A has full rank, at least one of $A \cup W_+$, $A \cup W_-$ is non-empty. Let us say that $A \cup W_+$ is non-empty. Let a be a point in $A \cup W_+$ that minimizes the distance to W. One then observes from the convexity and disjointness of W, W_-, W_+ that the midpoint sets $\frac{1}{2} \cdot 2(A \cap W)$, $\frac{1}{2} \cdot 2(A \cup W_+)$, $\frac{1}{2} \cdot (a + (A \cap W))$, and $\frac{1}{2} \cdot (2(A \cup W_-))$ are all disjoint. Since all these sets are contained in $\frac{1}{2} \cdot 2A$, we see that

$$|2(A \cap W)| + |2(A \cup W_+)| + |A \cap W| + |2(A \cup W_-)| \leq |2A| = K|A|.$$

Applying (5.13) we conclude (5.14).

Next, we apply the Freiman cube lemma to obtain a r-parallelepiped P with corners in A such that

$$|A \cap P| = \Omega_K(|A|). \tag{5.15}$$

Comparing this with (5.14) we see that P cannot be contained in a affine hyperplane (if ε is chosen sufficiently small). Since the parallelepiped P has $2r \leq 2K$ faces, each of which lies on an affine hyperplane, we thus see that, with $\mathrm{int}(P)$ denoting the interior of P, then

$$|A \cap \mathrm{int}(P)| \geq |A \cap P| - O(K\varepsilon|A|).$$

If Q denotes the 2^r corners of P, we observe that the sets $\{x + \mathrm{int}(P) : x \in Q\}$ are all disjoint; thus

$$|2(A \cap P)| \geq 2^r |A \cap \mathrm{int}(P)| \geq 2^r |A \cap P| - O(2^r K\varepsilon|A|). \tag{5.16}$$

The complement $V \backslash P$ of P in V can be partitioned into at $2r$ (unbounded) convex regions $B_1 \cup \cdots \cup B_{2r}$ (Exercise 5.2.4). Observe from convexity and disjointness that the midpoint sets $\frac{1}{2} \cdot 2(A \cap B_j)$ are disjoint from each other and from $2(A \cap P)$. Thus

$$|2A| \geq \sum_{j=1}^{2r} |2(A \cap B_j)| + |2(A \cap P)|.$$

Applying (5.14) we conclude

$$|2(A \cap P)| \leq K|A \cap P| + 2r\varepsilon|A|.$$

Combining this with (5.16), (5.15) and using the bound $r \leq K$ we see that

$$2^r \leq K + O_K(\varepsilon).$$

By choosing ε sufficiently small depending on $K < 2^d$ and d we obtain $r < d$ as desired. □

Exercises

5.2.1 [118] Show that Lemma 5.13 is still true if $A + A$ is replaced by $A - A$.

5.2.2 Let $d \geq 1$, $B := \{0, 1\}^d \subseteq \mathbf{R}^d$, and A be an additive subset of the convex hull of B (i.e. A lies in the solid unit cube $\{(x_1, \ldots, x_d) : 0 \leq x_1, \ldots, x_d \leq 1\}$. Show that

$$|A + B| \geq (\sqrt{2} - o_{d\to\infty}(1))^d |A|.$$

(Hint: reduce to the case where A is a subset of B, and then reduce further to the case where A consists of elements $(n_1, \ldots, n_d) \in \{0, 1\}^d$ where $n_1 + \cdots + n_d$ is fixed. Then restrict the elements of B in a similar manner and apply the covering principle and Stirling's formula (1.52). You may find working out the counterexample in the next exercise to be helpful.)

5.2.3 Show that the quantity $\sqrt{2}$ in Exercise 5.2.2 cannot be improved, by setting A equal to those elements $(n_1, \ldots, n_d) \in B$ such that $n_1 + \cdots + n_d = \lfloor \frac{d}{2} \rfloor$.

5.2.4 Let V be an r-dimensional vector space, and let P be a r-parallelepiped in V which is not contained in any hyperplane. Show that $V \backslash P$ is the union of $2r$ unbounded convex regions (not necessarily open).