5.4.5 Prove Proposition 5.35. (In addition to Theorem 5.33 and Theorem 5.20, you may use the rank reduction argument as in the proof of Theorem 5.33.)

5.4.6 Let A be a bounded non-empty open set in $\mathbf{R}^d$ such that $\mathrm{mes}(A + A) \leq K \mathrm{mes}(A)$. Show that $K \geq 2^d$, and that one has the containment $A \subseteq B + P$, where B is a ball and P is a progression of rank $O(K^{O(1)})$ and volume $O(\exp(K^{O(1)}) \mathrm{mes}(A)/\mathrm{mes}(B))$. (Hint: take B to be a ball contained in A. Now replace $\mathbf{R}^d$ with a lattice adapted to the scale of B.)

5.5 Universal ambient groups

In this section we fix the order k of Freiman homomorphisms and isomorphisms, and shall frequently omit the phrase "of order k".

It is possible for two additive sets to be Freiman isomorphic even though their ambient groups are very different. For instance, the additive sets $(\{1, 2, 3\}, \mathbf{Z}_6)$, $(\{1, 2, 3\}, \mathbf{Z}_7)$, and $(\{1, 2, 3\}, \mathbf{Z})$ are all Freiman isomorphic of order 2, despite the groups $\mathbf{Z}_6, \mathbf{Z}_7, \mathbf{Z}$ being different. On the other hand, the additive set $(\{1, 2, 3\}, \mathbf{Z}_3)$ is not Freiman isomorphic of order 2 to any of the above sets and has a quite different additive structure. It is natural to ask whether there is some universal ambient group that one can place an additive set in, after Freiman isomorphism. To phrase this more precisely, we introduce

Definition 5.37 (Universal ambient group) Let (A, Z) be an additive set, and let the order k of the Freiman homomorphisms be fixed. We say that Z is a *universal ambient group* (of order k) for the additive set A if, every Freiman homomorphism $\phi : (A, Z) \to (B, W)$ has a unique extension to a group homomorphism $\phi_{\mathrm{ext}} : Z \to W$ (thus $\phi_{\mathrm{ext}}(x) = \phi(x)$ for all $x \in A$). More generally, we say that an additive group Z' is a universal ambient group for (A, Z) if there exists an additive set (A', Z') which is Freiman isomorphic to (A, Z) such that Z' is a universal ambient group for A'; we then call (A', Z') an *embedding* of (A, Z) inside the ambient group Z'.

Examples 5.38 Let $k = 2$, and consider the additive set $(A, Z) = (\{1, 2, 3\}, \mathbf{Z}_7)$. The group $\mathbf{Z}_7$ is not a universal ambient group for $A = \{1, 2, 3\}$, as can be seen for instance by considering the Freiman homomorphism $\phi : A \to \mathbf{Z}$ defined by $\phi(1) = 1$, $\phi(2) = 2$, $\phi(3) = 3$. This homomorphism cannot extend to a group homomorphism on $\mathbf{Z}_7$, since 1 has order 7 in $\mathbf{Z}_7$ but has infinite order in $\mathbf{Z}$. Even if one replaces the ambient group $\mathbf{Z}_7$ with $\mathbf{Z}$, the additive set $(\{1, 2, 3\}, \mathbf{Z})$ is still not placed inside a universal ambient group, because the translation map $\phi(x) := x + 1$ is a Freiman homomorphism on $\{1, 2, 3\}$ but does not extend to a group homomorphism on $\mathbf{Z}$. On the other hand, the additive set $(\{(1, 1), (2, 1), (3, 1)\}, \mathbf{Z}^2)$ *is* placed inside a universal ambient group, as one can easily verify. But the additive set $(\{(1,1,0),(2,1,0),(3,1,0)\},\mathbf{Z}^3)$ is not placed inside a universal ambient group for a different reason, namely that the extension of Freiman homomorphisms to group homomorphisms is not unique (one has too much freedom to decide what to do with the third coordinate).

As stated, the definition of a universal ambient group is invariant under Freiman isomorphism. Also, if an additive set A has two universal ambient groups Z and Z', then they are necessarily group isomorphic (as can be seen by extending the obvious Freiman isomorphism between the two associated embeddings of A). Thus universal ambient groups, if they exist, are unique up to group isomorphism (for fixed k).

Lev and Konyagin [232] observed that universal ambient groups always exist:

Theorem 5.39 (Existence of universal ambient groups) *[232] Fix $k \geq 2$, and let (A, Z) be an additive set. Then there exists a universal ambient group Z' for A. Furthermore, if A' is an embedding of A inside this ambient group Z', then Z' is generated as a group by A'. In particular, Z' is finitely generated.*

Proof Let $\mathbf{Z}^A$ be a group of rank $|A|$ which is freely generated by some basis $\{e_a : a \in A\}$. Let $\langle X\rangle$ be the subgroup of $\mathbf{Z}^A$ generated by the elements

$$X := \{e_{a_1} + \cdots + e_{a_k} - e_{a'_1} - \cdots - e_{a'_k} : a_1, \ldots, a_k, a'_1, \ldots, a'_k \in A, a_1 + \cdots + a_k = a'_1 + \cdots + a'_k\}.$$

We then define $Z' := \mathbf{Z}^A/\langle X\rangle$, and let A' be the image of the basis $\{e_a : a \in A\}$ under the canonical quotient map $\pi : \mathbf{Z}^A \to \mathbf{Z}^A/\langle X\rangle$. It is clear that Z' is generated by A'. We now show that the map $\iota : A \to A'$ defined by $\iota(a) := \pi(e_a)$ is a Freiman isomorphism. Since this map is surjective, it suffices by Exercise 5.3.1 to show that

$$a_1 + \cdots + a_k = a'_1 + \cdots + a'_k \iff \iota(a_1) + \cdots + \iota(a_k) = \iota(a'_1) + \cdots + \iota(a'_k).$$

But this is clear from the construction of $\mathbf{Z}^A/\langle X\rangle$.

Next, let $\phi : (A', Z') \to (B, W)$ be a Freiman homomorphism. Let $\psi : \mathbf{Z}^A \to W$ be the unique group homomorphism such that $\psi(e_a) = \phi(\iota(a))$ for all $a \in A$; this is uniquely defined since the basis $\{e_a : a \in A\}$ freely generates $\mathbf{Z}^A$. Also it is clear that ψ annihilates X, and hence $\langle X\rangle$. Thus ψ descends to a group homomorphism $\phi_{\mathrm{ext}} : \mathbf{Z}^A/\langle X\rangle \to W$, and it is easily verified that ϕ_{ext} extends ϕ. This proves existence of extensions. To prove uniqueness, it suffices to show that any two group homomorphisms from Z' to W which agree on A' will agree on all of Z'. But this follows since Z' is generated by A'. □

For an alternative construction of the universal ambient group, see Exercise 5.5.1. For some examples of universal ambient groups, see Exercise 5.5.1 and Exercise 5.5.17.

If (A, Z) is an additive set with universal ambient group Z, then we can define a *degree map* $\deg : Z \to \mathbf{Z}$ to be the group homomorphism extending the trivial Freiman homomorphism $a \mapsto 1$. Thus deg equals 1 on A, equals 2 on $2A$, and more generally equals $l - m$ on $lA - mA$. Thus in the universal ambient group the sets nA for $n \in \mathbf{Z}$ are all disjoint. Also observe that deg must annihilate the torsion group $\mathrm{Tor}(Z) := \{x \in Z : nx = 0 \text{ for some } n \in \mathbf{Z}^+\}$ of Z, since the range $\mathbf{Z}$ of deg is torsion-free. This shows that $Z/\mathrm{Tor}(Z)$ is a non-trivial torsion-free additive group, and hence by Corollary 3.6 is group isomorphic to $\mathbf{Z}^{d+1}$ for some $d \geq 0$. Since all universal ambient groups are group isomorphic, this quantity d depends only on the additive set A, and we give it a name:

Definition 5.40 (Freiman dimension) Let A be an additive set. We define the *Freiman dimension* of A to be the unique non-negative integer $\dim(A) = d$ such that $Z/\mathrm{Tor}(Z)$ is group isomorphic to $\mathbf{Z}^{d+1}$ for every universal ambient group Z of A.

Note that the Freiman dimension depends on the choice k of the order of Freiman homomorphism; see Exercise 5.5.11. Traditionally one works with the Freiman dimension corresponding to the case $k = 2$. We caution that Freiman dimension is not monotone; again, see Exercise 5.5.11. The Freiman dimension can be interpreted as the largest rank that is attainable by a Freiman isomorphic copy of A in a vector space; see Exercise 5.5.10.

Let (A, Z) be an additive set with a universal ambient group Z, and let d be the Freiman dimension of A, and let Z be a universal ambient group for A. Then by Definition 5.40 we may identify $Z \equiv \mathbf{Z}^d \times \mathbf{Z} \times \mathrm{Tor}(Z)$; by applying a group isomorphism if necessary we may assume that the degree map $\deg : Z \to \mathbf{Z}$ corresponds to the $\mathbf{Z}$ coordinate of this identification, thus $\deg(n, m, x) = m$ for all $n \in \mathbf{Z}^d, m \in \mathbf{Z}, x \in \mathrm{Tor}(Z)$. Now let $\pi : Z \to \mathbf{Z}^d$ be the projection to the first factor. We call the additive set $[A] := (\pi(A), \mathbf{Z}^d)$ a *torsion-free universal representation* of A. It is easy to see that the torsion-free universal representation $[A]$ of an additive set A is unique up to affine group isomorphisms on $\mathbf{Z}^d$ (i.e. up to translations and elements of $SL_d(\mathbf{Z})$). Also, since A generates Z, we see that $\pi(A)$ must generate $\mathbf{Z}^{d+1}$, which implies that $\mathbf{Z}^d$ lies in the affine span of $[A]$. In other words, $\mathrm{rank}([A]) = d$.

Note that π induces a surjective Freiman homomorphism from A to $[A]$. If Z has no torsion group then this is in fact a Freiman isomorphism, but in general if A contains enough "torsion" then A and $[A]$ will not be Freiman isomorphic;

see Exercise 5.5.9. Nevertheless, $[A]$ remains a universal embedding of A in the category of embeddings into *torsion-free groups*. More precisely:

Proposition 5.41 *Let A be an additive set with Freiman dimension d, and let $[A] \subset \mathbf{Z}^d$ be a torsion-free universal representation of A. Let $\pi : A \to [A]$ be the associated Freiman homomorphism, and let $\phi : A \to (A', Z')$ be any Frieman homomorphism into a torsion-free additive group Z'. Then there exists a unique vector $v = (v_1, \ldots, v_d) \in (Z')^d$ and $a \in Z'$ such that $\phi(b) = a + \pi(b) \cdot v$ for all $b \in A$.*

Proof We may assume that A is embedded inside a universal ambient group $Z = \mathbf{R}^d \times \mathbf{R} \times \mathrm{Tor}(Z)$, and that $[A] = \pi(A)$ where $\pi : Z \to \mathbf{R}^d$ is the projection to the first factor. On the other hand, ϕ extends to a group homomorphism $\phi_{\mathrm{ext}} : \mathbf{R}^d \times \mathbf{R} \times \mathrm{Tor}(Z) \to Z'$. Since Z' is torsion-free, ϕ_{ext} must annihilate $\mathrm{Tor}(Z)$, and thus ϕ_{ext} must take the form $\phi_{\mathrm{ext}}(n, m, x) = n \cdot v + m \cdot a$ for all $n \in \mathbf{R}^d$, $m \in \mathbf{R}$, $x \in \mathrm{Tor}(Z)$, where $v \in (Z')^d$ and $a \in \mathbf{Z}$. Since A is a subset of $\mathbf{R}^d \times \{1\} \times \mathbf{Z}$ and $\pi(n, m, x) = 1$, we thus have $\phi(b) = \phi_{\mathrm{ext}}(b) = \pi(b) \cdot v + a$ for all $b \in A$, as desired. □

From this and Freiman's lemma we can obtain

Corollary 5.42 *Let $k \geq 2$, and let A be an additive set in a torsion-free additive group Z such that* $\min(|A + A|, |A - A|) \leq (d + 1)|A| - \frac{d(d+1)}{2}$ *for some integer* $K \geq 1$. *Then* $\dim(A) < d$.

Proof Let $[A] = \pi(A)$ be a torsion-free universal representation of A. By Proposition 5.41 we have a Freiman homomorphism from $[A]$ back to A, and hence A and $[A]$ are Freiman isomorphic. Hence we may without loss of generality work with $[A]$ instead of A. But then the claim follows from Lemma 5.13 (or Exercise 5.2.1), since $\mathrm{rank}([A]) = d$. □

Thus, in the torsion-free case at least, sets with small doubling necessarily have small Freiman dimension. A slightly weaker statement is true when A is not a torsion-free additive group:

Corollary 5.43 *Let $k \geq 2$, and let A be an additive set. Then* $\dim(A) < \sigma[A]^{O(1)}$.

Proof Let $K := \sigma[A]$ and $d := \dim(A)$. If $K < \frac{3}{2}$ then $d = 0$ (Exercise 5.5.13). Hence we may assume $K \geq \frac{3}{2}$, and it will now suffice to show $d = O(K^{O(1)})$. Without loss of generality we may assume that A is embedded in a universal ambient group Z. From Proposition 2.26 we see that $A + A$ can be covered by $O(K^{O(1)})$ translates of A. Applying the quotient map $\pi : Z \to Z/\mathrm{Tor}(Z) \equiv \mathbf{Z}^{d+1}$, we then see that $\pi(A) + \pi(A)$ can be covered by $O(K^{O(1)})$ copies of $\pi(A)$, and thus $|2\pi(A)| \leq K^{O(1)}|\pi(A)|$. But $\pi(A)$ is Freiman isomorphic to a torsion-free universal representation $[A]$ of A; thus $|2[A]| \leq K^{O(1)}|[A]|$. On the other hand, since $\mathrm{rank}([A]) = d$, we see from Lemma 5.13 that $|2[A]| > (d+1)|[A]| - \frac{d(d+1)}{2}$. Since $|[A]| \geq \mathrm{rank}(A) + 1 = d + 1$ (for instance), we thus have $|2[A]| > \frac{d}{2}|[A]|$ (for instance). Combining this with the upper bound on $|2[A]|$ we obtain the result. □

For a refinement of the bounds in this corollary, see Exercise 6.5.18.

Exercises

5.5.1 For any additive sets A, B, let $\mathrm{Hom}_k(A \to B)$ denote the space of Freiman homomorphisms (of order k) from A to B. Since A is an additive set, observe that $\mathrm{Hom}_k(A \to \mathbf{R}/\mathbf{Z})$ is an additive group which can be viewed as a compact subgroup of a torus. In particular it has a Pontryagin dual $Z' := \widehat{\mathrm{Hom}_k(A \to \mathbf{R}/\mathbf{Z})}$, defined as the space of all continuous group homomorphisms from $\mathrm{Hom}(A \to \mathbf{R}/\mathbf{Z})$ to the circle group $\mathbf{R}/\mathbf{Z}$. For any $a \in A$, define the *Gelfand transform* $\hat{a} \in Z'$ of a by the formula

$$\hat{a}(\chi) := \chi(a) \text{ for all } \chi \in \mathrm{Hom}_k(A \to \mathbf{R}/\mathbf{Z}),$$

and let $A' := \{\hat{a} : a \in A\}$. Show that (A', Z') is Freiman isomorphic to (A, Z), and that Z' is a universal ambient group for A.

5.5.2 Let A be an additive set. Show that $\mathbf{Z}^{|A|+1}$ is a universal ambient group for A if and only if A is a B_k set (see Definition 4.27), in which case the additive set $(\{e_j + e_{|A|+1} : 1 \leq j \leq |A|\}, \mathbf{Z}^{|A|+1})$ is an embedding of A into $\mathbf{Z}^{|A|+1}$. Here of course $e_1, \ldots, e_{|A|+1}$ is the standard basis for $\mathbf{Z}^{|A|+1}$.

5.5.3 Let A be an additive set, and let $\chi : A \to \mathbf{R}/\mathbf{Z}$ be a Freiman homomorphism. Let us say that χ is *infinitely divisible* if for every integer n there exists a Freiman homomorphism $\chi/n : A \to \mathbf{R}/\mathbf{Z}$ which, when multiplied by n, yields χ. Show that χ is infinitely divisible if and only if there exists a Freiman homomorphism $\phi : A \to \mathbf{R}$ such that $\phi \bmod 1 = \chi$. Conclude that the tangent space of the compact group $\mathrm{Hom}_k(A \to \mathbf{R}/\mathbf{Z})$ at the origin is canonically identifiable with $\mathrm{Hom}_k(A \to \mathbf{R})$.

5.5.4 Let $\phi : A \to B$ be a Freiman homomorphism (resp. isomorphism). Show that the map $\phi^\dagger : \mathrm{Hom}_k(B \to \mathbf{R}/\mathbf{Z}) \to \mathrm{Hom}_k(A \to \mathbf{R}/\mathbf{Z})$ defined by $\phi^\dagger(\chi) := \chi \circ \phi$ is a group homomorphism (resp. isomorphism). Also, if $\phi : A \to B$ and $\psi : B \to \mathbf{C}$ are Freiman homomorphisms, show that $(\phi \circ \psi)^\dagger = \psi^\dagger \circ \phi^\dagger$. Show that the *adjoint functor* $\phi \mapsto \phi^\dagger$ is a bijection between Freiman homomorphisms from A to B, and group homomorphisms from $\mathrm{Hom}_k(B \to \mathbf{R}/\mathbf{Z})$ to $\mathrm{Hom}_k(A \to \mathbf{R}/\mathbf{Z})$.

5.5.5 Let G be an additive set which is also an additive group (i.e. $G + G = G$). Show that $\mathrm{Hom}_k(G \to \mathbf{R}/\mathbf{Z})$ is canonically identifiable with $\hat{G} \times (\mathbf{R}/\mathbf{Z})$, where $\hat{G}$ is the Pontryagin dual of G, i.e. the space of group homomorphisms from G to $\mathbf{R}/\mathbf{Z}$. If A is an additive set contained in G, give examples to show that $\mathrm{Hom}_k(A \to \mathbf{R}/\mathbf{Z})$ can be much larger or much smaller than $\mathrm{Hom}_k(G \to \mathbf{R}/\mathbf{Z})$, although Freiman duality will convert the inclusion map from A to G to a group homomorphism from $\mathrm{Hom}_k(G \to \mathbf{R}/\mathbf{Z})$ to $\mathrm{Hom}_k(A \to \mathbf{R}/\mathbf{Z})$.

5.5.6 Let $k = 2$. Show that the universal ambient group of $A = (\{1, 2, 3\}, \mathbf{Z}_6)$ (or $(\{1, 2, 3\}, \mathbf{Z}_7)$, or $(\{1, 2, 3\}, \mathbf{Z})$) is canonically identifiable with $\mathbf{Z}^2$, with $\hat{A}$ being identified with $\{(1, 1), (2, 1), (3, 1)\}$. Show on the other hand that the universal ambient group of $A = (\{1, 2, 3\}, \mathbf{Z}_3)$ is canonically identified with $\mathbf{Z}_3 \times \mathbf{Z}$, with $\hat{A}$ identified with $\{(1, 1), (2, 1), (3, 1)\}$. Show that the universal ambient group of $A = (\{1, 2, 4, 5\}, \mathbf{Z})$ is canonically identifiable with $\mathbf{Z}^3$, with $\hat{A}$ being identified with $\{(0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 1, 1)\}$.

5.5.7 Let (A, Z) be an additive set embedded inside a universal ambient group Z, let (B, W) be another additive set, let $\phi : A \to B$ be a Freiman homomorphism, and let $\phi^{\mathrm{ext}} : Z \to W$ be the group homomorphism extension. Show that ϕ is a Freiman isomorphism if and only if the kernel $\ker(\phi^{\mathrm{ext}}) := \{x \in Z : \phi^{\mathrm{ext}}(x) = 0\}$ of ϕ^{ext} is disjoint from $(kA - kA)\backslash\{0\}$, or equivalently if ϕ^{ext} is injective on kA.

5.5.8 Let (A, Z) be an additive set embedded inside a universal ambient group Z, and let G be an additive group. Show that G contains a subset A' that is Freiman isomorphic to A if and only if G contains a subgroup H that is group isomorphic to Z/Γ for some subgroup Γ of Z which is disjoint from $(kA - kA)\backslash\{0\}$.

5.5.9 Let (A, Z) be an additive set embedded inside a universal ambient group Z. Show that A and $[A]$ are Freiman isomorphic if and only if $(kA - kA) \cap \mathrm{Tor}(Z) = \{0\}$. Note from Proposition 5.41 that A can be embedded into a torsion-free additive group if and only if A and $[A]$ are Freiman isomorphic.

5.5.10 Let A be an additive set in a torsion-free additive group Z. Show that there exists a Freiman-isomorphic copy (A', V') of (A, Z) inside a vector space V' such that $\mathrm{rank}(A') = \dim(A)$. Furthermore, we have $\mathrm{rank}(A'') \leq \dim(A)$ for any other Freiman isomorphic copy (A'', V'') of (A, Z) in a vector space.

5.5.11 Let (A, Z) be the additive set $(\{1, 2, 4, 5\}, \mathbf{Z})$. Show that $\dim(A) = 4$ if $k = 1$, that $\dim(A) = 2$ if $k = 2$, and $\dim(A) = 1$ for $k \geq 3$. In particular,