when $k = 2$, conclude that $\dim(\{1, 2, 4, 5\}) > \dim(\{1, 2, 3, 4, 5\})$, thus demonstrating that Freiman dimension is not monotone.

5.5.12 Show that the Freiman dimension $\dim(A) = \dim_k(A)$ of an additive set is a non-increasing function of k, thus $\dim_{k+1}(A) \leq \dim_k(A)$.

5.5.13 Let $k \geq 2$, and let A be an additive set such that $\sigma[A] < \frac{3}{2}$. Show that $\dim(A) = 1$. (Hint: embed A in a universal ambient group and apply Corollary 5.6.)

5.5.14 Let (A, Z) and (A', Z') be additive sets. Show that $\dim(A \oplus A') = \dim(A) + \dim(A')$.

5.5.15 Let $\phi : A \to A'$ be a surjective Freiman homomorphism. Show that $\dim(A') \leq \dim(A)$.

5.5.16 Let A be an additive set, and let Z be a universal ambient group for A. Show that $\mathrm{Tor}(Z) = \{0\}$ if and only if the group homomorphism $\pi : \mathrm{Hom}_k(A \to \mathbf{R}) \to \mathrm{Hom}_k(A \to \mathbf{R}/\mathbf{Z})$ defined by $\pi(\phi) := \phi \bmod 1$ is surjective, or in other words every Freiman homomorphism from A to $\mathbf{R}/\mathbf{Z}$ lifts up to a Freiman homomorphism from A to $\mathbf{R}$.

5.5.17 Let $k = 2$ and consider the set $A := \{2e_1, e_1 + e_2, 2e_2, e_2 + e_3, 2e_3, e_3 + e_4, 2e_4, e_4 + e_1\}$ in $\mathbf{Z}^4$, where e_1, e_2, e_3, e_4 is the standard basis; one can view this as a generic skew quadrilateral together with the midpoints. Show that $(A, \mathbf{Z}^4)$ has $\mathbf{Z}^4 \times (\mathbf{Z}/2\mathbf{Z})$ as a universal ambient group. Thus it is possible for the universal ambient group to contain some torsion even when the additive set can be embedded in a torsion-free additive group. Write down an embedding of A in the universal ambient group $\mathbf{Z}^2 \times (\mathbf{Z}/2\mathbf{Z})$, and compare it with a torsion-free representation $[A]$ of A; are they Freiman isomorphic to each other?

5.5.18 Generalize Theorem 5.11 to handle additive sets A in any torsion-free additive group.

5.6 Freiman's theorem in an arbitrary group

Now we use the universal group, combined with Fourier analysis and additive geometry, to obtain Freiman's theorem in an arbitrary additive group. This result was first obtained by Green and Ruzsa [157]; the approach here is inspired by their argument but is arranged somewhat differently, relying in particular on volume bounds on polar bodies instead of the Ruzsa–Chang theorem (Theorem 5.30), and working in the universal ambient group rather than by introducing a sequence of successively smaller ambient groups to contain the additive set A.

Observe that in some inverse sum set theorems (Corollary 5.6, Theorem 5.27) a set with small doubling was contained inside a finite group (or a coset of such a group), whereas in other inverse sum set theorems (Theorem 5.11, Theorem 5.32, and to a lesser extent Corollary 5.19) a set with small doubling was contained inside a progression. In general, it is convenient to place a set of small doubling inside a *coset progression* $P + H$, which was defined in Definition 4.21.

Theorem 5.44 (Freiman's theorem in an arbitrary group) *[157] Let $K \geq 1$, and let (A, Z) be an additive set in an arbitrary group Z such that $|A + A| \leq K|A|$. Then there a coset progression $P + H$ of rank at most* $\dim(A)$ *such that $A \subseteq P + H$ and $|P||H| \leq \exp(O(K^{O(1)}))|A|$. If Z is the universal ambient group of A, then we can take $H = \mathrm{Tor}(Z)$.*

One can make the constants in $\exp(O(K^{O(1)}))$ more explicit; see [157].

Proof Here we shall fix the order k of the Freiman homomorphisms under consideration to be $k = 2$. Without loss of generality we may assume Z is the universal ambient group; the general case then follows from Definition 5.37 (and the observation that the image of a group or progression under a group homomorphism is still a group or progression). We write $d := \dim(A)$; from Corollary 5.43 we have $d = O(K^{O(1)})$.

We know that Z is isomorphic to $\mathbf{Z}^d \times \mathbf{Z} \times \mathrm{Tor}(Z)$; we shall abuse notation and *identify* Z with $\mathbf{Z}^d \times \mathbf{Z} \times \mathrm{Tor}(Z)$, in particular identifying $\mathrm{Tor}(Z)$ with $\{(0, 0)\} \times \mathrm{Tor}(Z)$. We can also arrange matters so that the $\mathbf{Z}$ component of Z is given by the degree map, thus $\deg((n, m, x)) = m$ for all $n \in \mathbf{Z}^d$, $m \in \mathbf{Z}$, $x \in \mathrm{Tor}(Z)$, and A lives entirely in $\mathbf{Z}^d \times \{1\} \times \mathrm{Tor}(Z)$. By using a group isomorphism to translate A in the $\mathbf{Z}^d \times \mathrm{Tor}(Z)$ direction if necessary, we may assume that $(0, 1, 0) \in A$.

At present, Z is not a finite group and so we cannot directly apply the Fourier analytic techniques from Chapter 4. Thus we shall truncate Z to a finite group (cf. the use of Lemma 5.26 to prove Theorem 5.30); an alternative approach (which we do not pursue here, due to some minor measure-theoretic and analytic issues which arise) is to extend the theory of the Fourier transform and of Chapter 4 to infinite additive groups. We choose an extremely large prime number p depending on A (much larger than any of the $d + 1$ coefficients of elements of A in the $\mathbf{Z}^{d+1}$ component of Z), and let $\pi_p : Z \to Z_p$ be the canonical projection from $Z = \mathbf{Z}^d \times \mathbf{Z} \times \mathrm{Tor}(Z)$ to the finite additive group $Z_p := \mathbf{Z}_p^d \times \mathrm{Tor}(Z)$. If p is sufficiently large, then π_p is a Freiman isomorphism from A to the additive set $A_p := \pi_p(A)$. We endow Z_p with the symmetric non-degenerate bilinear form

$$(\xi, \eta) \cdot (x, y) = \frac{x\xi}{p} + \eta \cdot y$$

for all $x, \xi \in \mathbf{Z}_p^d$ and $y, \eta \in \mathrm{Tor}(Z)$, where $\eta \cdot y$ is some symmetric non-degenerate bilinear form on $\mathrm{Tor}(Z)$ (the exact choice of which will be irrelevant).

Let $\alpha := 1 - \frac{1}{10^5 K^2}$. Now we establish some lower bounds on the spectrum $\mathrm{Spec}_\alpha(A_p - A_p)$ of $A_p - A_p$, as defined in Definition 4.34. □

Lemma 5.45 *We have* $|\mathrm{Spec}_\alpha(A_p - A_p)| \geq \exp(-O(K^{O(1)}))|Z_p|/|A_p|$.

Proof We first control the size of sum sets nA for very large n. Since A_p is Freiman isomorphic to A, we have $\sigma[A_p] \leq K$. By Proposition 2.26 we can thus contain A_p inside a translate of a K^C-approximate group H of size $|H| \leq K^C|A_p|$; thus $2H \subseteq H + X$ for some X of cardinality $O(K^{O(1)})$. Iterating this we see that $nH \subseteq H + (n-1)X$, and thus

$$\begin{aligned} |n(A_p - A_p)| &\leq |2nH| \\ &\leq |H||(2n-1)X| \\ &\leq K^{O(1)}|A_p|\binom{|X| + 2n - 2}{|X|} \\ &\leq K^{O(1)}|A_p|(|X| + 2n - 2)|X|. \end{aligned}$$

If we then set $n := CK^C$ for a sufficiently large constant C, we can ensure that

$$|n(A_p - A_p)| \leq \frac{1}{2}\alpha^{2-2n}|A_p - A_p|.$$

We then apply Lemma 4.38 to obtain

$$|\mathrm{Spec}_\alpha(A_p - A_p)|\, \mathbf{P}_Z(A_p - A_p) \geq \frac{1}{2}\alpha^{2-2n}$$

and the claim follows (recall $|A_p - A_p| \leq K^2|A_p|$ from Ruzsa's triangle inequality). □

Now we can use the theory of Freiman homomorphisms and the universal ambient group to eliminate the role of the torsion group. Let $\Pi : Z \to \mathbf{Z}^d \subset \mathbf{R}^d$ be the canonical projection from $Z = \mathbf{Z}^d \times \mathbf{Z} \times \mathrm{Tor}(Z)$ to $\mathbf{Z}^{d+1}$, thus $\Pi(A)$ is a subset of $\mathbf{Z}^{d+1}$ and hence of $\mathbf{R}^{d+1}$.

Lemma 5.46 *We have* $\mathrm{Spec}_\alpha(A_p - A_p) \subseteq \mathbf{Z}_p^d \times \{0\}$. *Furthermore, if* $\xi' \in \mathbf{Z}_p^d$ *is such that* $(\xi', 0) \in \mathrm{Spec}_\alpha(A_p - A_p)$, *then there exists* $\tilde{\xi} \in \frac{1}{p} \cdot \mathbf{Z}^d \subset \mathbf{R}^d$ *with* $\tilde{\xi} = \xi'/p \pmod 1$ *such that* $|\langle x, \tilde{\xi}\rangle| \leq \frac{1}{5}$ *for all* $x \in \Pi(A) - \Pi(A)$.

Proof From Ruzsa's triangle inequality we have $|A_p - A_p| \leq K^2|A_p|$. From Proposition 4.40 we thus see that $A_p - A_p \subseteq \mathrm{Bohr}_Z(\mathrm{Spec}_\alpha(A_p - A_p), \frac{1}{50})$. Thus if $\xi \in \mathrm{Spec}_\alpha(A_p - A_p)$, then $|e(\xi \cdot x) - 1| \leq \frac{1}{50}$ for all $x \in A_p - A_p$. In particular we can find a phase $e^{2\pi i\theta}$ for some $\theta \in \mathbf{R}$ such that $|e(\xi \cdot x) - e^{2\pi i\theta}| \leq \frac{1}{50}$ for all

$x \in A_p$. We can thus find a function $\chi : A_p \to \mathbf{R}$ such that $e(\xi \cdot x) = e(\chi(x))$ and $\theta - \frac{1}{10} < \chi(x) < \theta + \frac{1}{10}$ for all $x \in A_p$. It is then easy to see that $\chi : A_p \to \mathbf{R}$ is a Freiman homomorphism, and hence $\chi \circ \pi_p : A \to \mathbf{R}$ is a Freiman homomorphism. Since Z is a universal ambient group for A, we thus see that we can extend $\chi \circ \pi_p$ to a group homomorphism $(\chi \circ \pi)_{\mathrm{ext}} : Z \to \mathbf{R}$. But since $\mathbf{R}$ is torsion-free, this group homomorphism must annihilate the torsion group $\mathrm{Tor}(Z)$. In particular, the map $\phi : x \mapsto (\chi \circ \pi_p)_{\mathrm{ext}}(x) \bmod 1$ is a group homomorphism from Z to $\mathbf{R}/\mathbf{Z}$ which annihilates $\mathrm{Tor}(Z)$. On the other hand, the map $\tilde{\phi} : x \mapsto \xi \cdot \pi_p(x)$ is another group homomorphism from Z to $\mathbf{R}/\mathbf{Z}$ which agrees with ϕ on A. Since Z is a universal ambient group for A, this means that $\phi = \tilde{\phi}$, and thus $\tilde{\phi}$ must also annihilate $\mathrm{Tor}(Z)$. In other words we see that $\xi \cdot x = 0$ whenever $x \in \mathrm{Tor}(Z)$, which means that $\xi \in \mathbf{Z}_p^d \times \{0\}$, and the first claim follows.

Now let $\xi' \in \mathbf{Z}_p^d$ be such that $(\xi', 0) \in \mathrm{Spec}_\alpha(A_p - A_p)$. Then as before we can find a Freiman homomorphism $\chi : A_p \to \mathbf{R}$ such that

$$(\xi', 0) \cdot x = \chi(x) \bmod 1 \text{ for all } x \in A_p \tag{5.17}$$

and a $\theta \in \mathbf{R}$ such that

$$\theta - \frac{1}{10} < \chi(x) < \theta + \frac{1}{10} \text{ for all } x \in A_p - A_p, \tag{5.18}$$

and we have a group homomorphism $(\chi \circ \pi)_{\mathrm{ext}} : Z \to \mathbf{R}$ which extends $\chi \circ \pi$ and annihilates $\mathrm{Tor}(Z)$. Since $Z = \mathbf{Z}^d \times \mathbf{Z} \times \mathrm{Tor}(Z)$, we thus see that there exist $\tilde{\xi} \in \mathbf{R}^d$ and $\eta \in \mathbf{R}$ such that

$$(\chi \circ \pi)_{\mathrm{ext}}(n, m, x) = n \cdot \tilde{\xi} + m\eta \text{ for all } n \in \mathbf{Z}^d, m \in \mathbf{Z}, x \in \mathrm{Tor}(Z).$$

Restricting this to elements of A (which lie in $\mathbf{Z}^d \times \{1\} \times \mathrm{Tor}(Z)$, we obtain

$$\chi((n \bmod p, x)) = \chi(\pi(n, 1, x)) = n \cdot \tilde{\xi} + \eta \text{ whenever } (n, 1, x) \in A. \tag{5.19}$$

Applying (5.17) we obtain

$$n \cdot \xi'/p = n \cdot \tilde{\xi} + \eta \pmod 1 \text{ whenever } (n, 1, x) \in A.$$

Since $(0, 1, 0) \in A$, we conclude that $\eta = 0 \pmod 1$. Since A generates all of $Z = \mathbf{Z}^d \times \mathbf{Z} \times \mathrm{Tor}(Z)$, we infer that $\tilde{\xi} = \xi'/p \pmod 1$ as desired; in particular $\tilde{\xi} \in \frac{1}{p} \cdot \mathbf{Z}^d$. Next, we apply (5.18) to deduce that

$$\theta - \frac{1}{10} < n \cdot \tilde{\xi} + \eta < \theta + \frac{1}{10} \text{ whenever } (n, 1, x) \in A$$

and thus

$$|(n - n') \cdot \tilde{\xi}| < \frac{1}{5} \text{ whenever } n, n' \in \Pi(A),$$

and the claim follows (note that the dot product $n \cdot x$ and the inner product $\langle n, x\rangle$ agree when $n \in \mathbf{Z}^d$ and $x \in \mathbf{R}^d$). $\square$

Since $\Pi(A) - \Pi(A)$ is a subset of $\mathbf{Z}^d$, it is also a subset of $\mathbf{R}^d$. Let B be the convex body generated by the open convex hull of $\Pi(A) - \Pi(A)$; note that B is open and non-empty because A generates Z, and hence $\Pi(A)$ generates $\mathbf{Z}^d$. Introducing the polar body

$$B^\circ := \{x \in \mathbf{R}^d : |x \cdot y| < 1 \text{ for all } y \in B\}$$

of B, we can rewrite the conclusion of Lemma 5.46 as

$$\tilde{\xi} \in \frac{1}{5} \cdot B^\circ.$$

Combining this with Lemma 5.45, we thus see that

$$\left|\left(\frac{1}{5} \cdot B^\circ\right) \cap \left(\frac{1}{p} \cdot \mathbf{Z}^d\right)\right| \geq \frac{\exp(-CK^C)|Z_p|}{|A_p|} = \frac{\exp(-O(K^{O(1)}))p^d|\mathrm{Tor}(Z)|}{|A|}$$

and thus

$$p^{-d}\left|B^\circ \cap \left(\frac{1}{p} \cdot \mathbf{Z}^d\right)\right| \geq \frac{\exp(-CK^C)|\mathrm{Tor}(Z)|}{|A|}.$$

Now we take limits as $p \to \infty$. Since B° is open and bounded, the left-hand side is just the Riemann sum for $\mathrm{mes}(B^\circ)$, and thus

$$\mathrm{mes}(B^\circ) \geq \exp\big(-O\big(K^{O(1)}\big)\big)|\mathrm{Tor}(Z)|/|A|.$$

Now we use the machinery from Chapter 3. Using the rather crude bound

$$\mathrm{mes}(B^\circ)\mathrm{mes}(B) \leq O(1)^d = O(1)^{K^{O(1)}} \tag{5.20}$$

(see Exercise 5.6.1), we can convert this lower bound on B° to an upper bound for B:

$$\mathrm{mes}(B) \leq \exp\big(O\big(K^{O(1)}\big)\big)|A|/|\mathrm{Tor}(Z)|.$$

Note that $B \cap \mathbf{Z}^d$ contains $\Pi(A) - \Pi(A)$; since $\Pi(A)$ generates $\mathbf{Z}^d$, we thus conclude that $B \cap \mathbf{Z}^d$ linearly spans $\mathbf{R}^d$. From this and Lemma 3.26 we see that

$$|B \cap \mathbf{Z}^d| \leq \exp\big(O\big(K^{O(1)}\big)\big)|A|/|\mathrm{Tor}(Z)|$$

where we have used the earlier observation $d = O(K^{O(1)})$ to absorb the $3^d d!/2^d$ factor from that Lemma. Applying the discrete John theorem (Lemma 3.36) we can thus place B inside a progression $Q \subseteq \mathbf{Z}^d$ of rank at most d and volume

$$|Q| \leq \exp\big(O\big(K^{O(1)}\big)\big)|A|/|\mathrm{Tor}(Z)|,$$

again using the observation $d = O(K^{O(1)})$, this time to absorb the factors of $(d^{2d})^d$ that will appear. Since A was normalized to contain $(0, 1, 0)$, we have the inclusions $\Pi(A) \subseteq \Pi(A) - \Pi(A) \subseteq B \cap \mathbf{Z}^d \subseteq Q$, and hence $A \subseteq \Pi^{-1}(Q)$. But we may write $\Pi^{-1}(Q) = P + G$ where P is an isomorphic copy of Q, and $G := \mathrm{Tor}(Z)$. Theorem 5.44 follows.

Remark 5.47 It seems of interest to improve the exponential losses $\exp(O(K^{O(1)}))$ in the above argument. Many of these losses are really exponential in the Freiman dimension d rather than in the doubling constant K, so one expects to gain somewhat when the Freiman dimension is small. However, the main step where the exponential losses are largest lies in the proof of Lemma 5.45, where one is forced to control extremely large sum sets of A_p in order to obtain a lower bound on the size of the spectrum. It may be that one will have to use a non-Fourier-analytic approach in order to avoid this type of loss. On the other hand, the asymptotic behavior of iterated sum sets is certainly relevant to the task of containing A inside a convex body or arithmetic progression (see Exercise 5.6.4). However, it may well be that this type of argument can at least be pushed to improve $\exp(O(K^{O(1)}))$ to a factor like $\exp(O(K \log^{O(1)} K))$ or perhaps even $\exp(O(K))$.

We now comment briefly on the slightly different argument of Green and Ruzsa [157] in establishing the above theorem. Instead of working in a universal ambient group, which could be infinite, they proceed by first using a Freiman isomorphism (of order at least 16, say) to embed A inside a very large finite group (similar to the group Z_p used in the analysis here), and then to use an estimate similar to Lemmas 5.45 and 5.46 to reduce the size of this ambient group Z iteratively until $|Z| \leq \exp(CK^C)|A|$ (the point being that if $|Z| > \exp(CK^C)|A|$, then the arguments of Lemmas 5.45 and 5.46 can be used to locate a narrow Bohr set that contains A, which is then Freiman isomorphic to a subset of a smaller group than Z. At this point one can apply an extension of Theorem 4.42 (for arbitrary finite additive groups, not necessarily cyclic) to show that $2A - 2A$ contains the sum of a large progression and a large group, at which point one can conclude a Ruzsa–Chang type theorem for arbitrary groups, which then implies the above theorem by an argument similar to how Theorem 5.30 implies Theorem 5.32. In particular, they establish

Theorem 5.48 (Ruzsa–Chang theorem in arbitrary groups) *[157] Let A be an additive set in an arbitrary additive group Z such that $|A + A| \leq K|A|$ for some $K \geq 1$. Then $2A - 2A$ contains a set of the form $P + G$ where P is a proper symmetric progression of rank at most $CK(1 + \log K)$ and G is a finite subgroup of Z such that $|P + G| = |P||G| \geq e^{-CK(1+\log^2 K)}|A|$.*

Exercises

5.6.1 Let B be a symmetric convex body, and consider the Euclidean Fourier transform

$$\widehat{1_B}(\xi) := \int_{\mathbf{R}^d} 1_B(x) e(-\xi \cdot x)\, d\xi.$$

Show that this Fourier transform is large on a large subset of the polar body B°, and use this and the Plancherel theorem on $\mathbf{R}^d$ to establish (5.20). (A much sharper inequality than (5.6.1) is available, namely *Santalo's inequality* [306], but we will not need this inequality here.)

5.6.2 [157] Let A be an additive set with $|A + A| \leq K|A|$. Show that there exists a finite group Z of order $|Z| \leq \exp(O(K^{O(1)}))|A|$ such that A is Freiman isomorphic of order 2 (say) to a subset of Z. (Hint: combine the analysis of this section with Exercise 5.5.8.)

5.6.3 [154] Suppose p is a prime number, and A is an additive set in $\mathbf{Z}_p$ such that $|A + A| \leq K|A|$ for some $K \geq 1$. Suppose also that $|A| \leq \exp(-O(K^{O(1)}))p$ for some sufficiently large absolute constant $C > 1$. Show that A is Freiman isomorphic of order 2 to a subset of the integers $\mathbf{Z}$. This is known as the *Freiman rectification principle*; see [29], [154] for further discussion.

5.6.4 Let A be an additive set in $\mathbf{Z}^d$ which generates $\mathbf{Z}^d$, and let B be the convex hull of A. Show that $|nA| = (1 + o_{n\to\infty}(1))n^d \operatorname{mes}(B)$ as $n \to \infty$. (See [261] for more precise results of this type.)