

## 6.1 Basic Notions

A graph  $G = G(V, E)$  consists of a finite set  $V$  of *vertices* (*points*, *nodes*) and a finite set  $E$  of *edges*, where each edge is an unordered pair  $\{a, b\}$  of distinct vertices (thus we do not allow loops).

If  $\{a, b\} \in E$ , we say that the two vertices  $a$  and  $b$  are *adjacent* or *neighbors*. The collection of all the neighbors of  $a$  shall be denoted  $N(a)$ . The cardinality of  $N(a)$  is called the *degree* of  $a$  and is denoted  $\deg(a)$ .

Consider a subset  $V'$  of  $V$ . We refer to the graph  $G' = G'(V', E')$  where  $E' := \{e \in E : e \subseteq V'\}$  as the *induced* subgraph of  $G$  which is *spanned* by  $V'$ . A set  $V' \subset V$  is *independent* if it spans an empty graph, i.e., there is no edge with both endpoints in  $V'$ .

We say that the vertices  $a_0, \dots, a_k$  form a *path* of length  $k$  if  $\{a_i, a_{i+1}\}$  is an edge for all  $0 \leq i \leq k-1$ . If  $a_k = a_0$ , we refer to the path as a *cycle*. Three vertices  $a, b, c$  form a *triangle* if they form a cycle of length 3, i.e.  $\{a, b\}$ ,  $\{b, c\}$  and  $\{c, a\}$  are edges.

A graph is *bipartite* if one can partition its vertex set into two disjoint sets  $A$  and  $B$  so that every edge has one end point in  $A$  and another in  $B$ ;  $A$  and  $B$  are called the *color classes* of  $G$ . Bipartite graphs play an important role in what follows and when dealing with them, we prefer to use the notation  $G(A, B, E)$  instead of  $G(V, E)$ . Note that in a bipartite graph  $G = G(A, B, E)$ , two vertices in the same color class can only be connected by paths of even length, while vertices in opposite color classes can only be connected by paths of odd length. In particular all cycles must be of even length.

### Exercises

- 6.1.1 Prove that a graph  $G$  is bipartite if and only if all cycles are of even length.
- 6.1.2 Let  $A$  be a symmetric additive set (so  $A = -A$ ) in a finite additive group  $Z$ . The *Cayley graph* of  $A$  is defined to be the graph with vertex set  $Z$ , and two vertices  $x, y$  connected by an edge if and only if  $x - y \in A$ . Show that  $\deg(v) = |A|$  for all  $v \in Z$ , and that two points  $v, w \in Z$  are connected by a path of length  $n$  if and only if  $v - w \in n(A)$ . Show that  $G$  is connected if and only if  $A$  spans  $Z$ .
- 6.1.3 (Popularity principle for bipartite graphs) Let  $G(V_1, V_2, E)$  be a bipartite graph with  $V_2$  non-empty. Show that there exists a bipartite subgraph  $G'(V_1, V'_2, E')$  of  $G(V_1, V_2, E)$  with  $|E'| \geq |E|/2$  and  $\deg_{G'}(v_2) \geq |E|/2|V_2|$  for all  $v_2 \in V'_2$ .
- 6.1.4 (Cauchy–Schwarz for bipartite graphs) Let  $G(V_1, V_2, E)$  be a bipartite graph with  $V_1, V_2$  non-empty. Show that  $G$  contains at least  $|E|^2/|V_2|$