

such that $\deg_{G'}(a) \geq |B|/2K$ and $\deg_G(b) \geq |A|/2K$ for all $a \in A'$ and $b \in B'$. (Hint: choose A', B' to maximize the quantity $\frac{|E \cap (A' \times B')|}{|A'|^{1/2}|B'|^{1/2}}$.)

6.4.7 Prove Claim 6.22. (Hint: use induction.)

6.4.8 Using the same hypotheses as Theorem 2.29, show that for any $\varepsilon > 0$ there exists a set $G' \subseteq A' \times A'$ such that $|G'| \geq (1 - \varepsilon)|A'|^2$ and $|A' \overset{G'}{-} A'| \leq \frac{2(KK')^2}{\varepsilon}|A'|$.

6.4.9 Improve the 2^{12} factor in Theorem 2.29 to 2^{10} by exploiting the fact that all of the paths of length three constructed in Corollary 6.20 pass through \tilde{A}' , which is a slightly smaller set than A .

6.4.10 [38] Let A, B be additive sets in an ambient group Z , and let $G \subset A \times B$ be such that $|G| \geq |A||B|/K$ and $|A \overset{G}{+} B| \leq K|A|^{1/2}|B|^{1/2}$ for some $K \geq 1$. Show that there exist subsets A', B' of A, B such that $|A'| = \Omega(K^{-O(1)}|A|)$, $|B'| \geq \Omega(K^{-O(1)}|B|)$, $d(A', B') = O(1 + \log K)$, and $|G \cap (A' \times B')| = \Omega(K^{-O(1)}|A||B|)$. (Hint: the novelty here is that we still wish the refinement $A' \times B'$ to capture a large portion of G . This requires that one revisit the arguments in Lemma 6.19 and Corollary 6.20 and perform some additional “popularity” refinements to ensure that every time one reduces the size of A or B , one still keeps a significant fraction of elements from G . One may also need to use Lemma 2.30 at times to ensure that one also keeps a large number of “popular differences” between various refinements of A and B .) For an earlier result of this type, see [223].

6.5 Plünnecke's theorem

One of the most useful tools for the study of sum sets is Plünnecke's theorem. In order to state this theorem, we first need some notation.

Definition 6.23 (Magnification ratio) A *directed bipartite graph* is a triple $G(A, B, E)$, where A, B are finite sets (not necessarily disjoint) and $E \subset A \times B$ is a collection of pairs (a, b) from A and B . We write $G : A \rightarrow B$ to emphasize the directed nature of this graph, and also write $a \mapsto_G b$ to denote the statement that $(a, b) \in E$. If $X \subset A$, we use $G(X) := \{b \in B : a \mapsto b \text{ for some } a \in X\}$ to denote the image of X , and then define the *magnification ratio* $\|G\|$ of G to be the quantity

$$\|G\| = \min_{X \subset A: X \neq \emptyset} \frac{|G(X)|}{|X|}.$$

Equivalently, $\|G\|$ is the smallest number such that $|G(X)| \geq \|G\||X|$ for all sets $X \subseteq A$.

If $G : A \rightarrow B$ and $H : B \rightarrow C$ are two directed bipartite graphs, with A, B, C disjoint, we define the *composition* $H \circ G : A \rightarrow C$ to be the directed graph defined by setting $a \mapsto_{H \circ G} c$ in $H \circ G$ if and only if there exists $b \in B$ such that $a \mapsto_G b \mapsto_H c$.

One can also view a directed bipartite graph $G : A \rightarrow B$ as a multiply-valued function from A to B , and the magnification ratio is then a measure of the multiplicity of this function.

Example 6.24 Let A, B be additive sets with common ambient group. Then we can form the directed bipartite graph $G_{A,B} : A \rightarrow A + B$ by setting $a \mapsto_{G_{A,B}} a + b$ if and only if $a \in A$ and $b \in B$. Observe that

$$\|G_{A,B}\| := \min_{X \subseteq A: X \neq \emptyset} \frac{|X + B|}{|X|} \leq \frac{|A + B|}{|A|}.$$

Also, observe that if A, B, C are additive sets with $A, A + B, A + B + C$ disjoint, then $G_{A+B,C} \circ G_{A,B} = G_{A,B+C}$.

For general directed bipartite graphs one has the inequality $\|H \circ G\| \leq \|G\|\|H\|$. However there is a deeper inequality available for certain families of directed bipartite graphs known as *Plünnecke graphs*. While this concept can be given for abstract graphs, it is easiest to describe for graphs whose vertices lie in an additive group (which is always the case for our applications).

Definition 6.25 (Plünnecke graphs) Let A_0, A_1, A_2 be three additive sets in an additive group Z . Two directed bipartite graphs $G_1 : A_0 \rightarrow A_1$ and $G_2 : A_1 \rightarrow A_2$ are said to be *commutative* if, whenever $a, b, c \in Z$ are such that $a \mapsto_{G_1} a + b \mapsto_{G_2} a + b + c$ in G_2 , then one also has $a \mapsto_{G_1} a + c \mapsto_{G_2} a + b + c$. More generally, if $k \geq 2$, and A_0, \dots, A_k are additive sets in Z , we define a *Plünnecke graph* of order k to be a k -tuple (G_1, \dots, G_k) of bipartite graphs $G_j : A_{j-1} \rightarrow A_j$ such that each adjacent pair G_j, G_{j+1} for $1 \leq j < k$ is commutative.

Here is a more informal way to describe commutativity: if two adjacent edges of a parallelogram lie in $G_1 \cup G_2$, then so do the other two edges of the parallelogram).

Example 6.26 Let A, B be additive sets. Then the k -tuple

$$(G_{A,B}, G_{A+B,B}, \dots, G_{A+(k-1)B,B})$$

of directed bipartite graphs (as defined in Example 6.24) forms a Plünnecke graph.

We are now ready to state Plünnecke’s theorem.