In this chapter, we present two different approaches. The first is the combinatorial approach of Erdős and later authors, which phrases the problem in the theory of *set systems* (collections of subsets of a given set), thus allowing one to apply the theory of extremal set systems. This approach is very elegant and gives sharp results, but it is difficult to extend it to cases in which one has more complicated constraints on the steps v_j. The second, and rather different approach, is the Fourier-analytic one introduced by Halász. The bounds obtained by this approach are usually off by an absolute constant from the best possible results, but the arguments are more flexible.

A general theme will be that strong concentration or repetition of the above sums is closely related to strong additive structure among the steps $v_1, \ldots, v_n$. At one extreme, if the group Z has no 2-torsion, then all the sums are distinct if and only if the $v_1, \ldots, v_n$ are dissociated (see Definition 4.32). At another extreme, if the $v_1, \ldots, v_n$ are contained inside an arithmetic progression of small rank and volume, then one expects plenty of repetitions among the sums. The situation is thus somewhat analogous to the theory of sum set estimates and inverse sum set theorems studied in previous chapters, and indeed there will be strong similarities in our treatment of the two (in particular, the parallel use of combinatorial and Fourier-analytic methods).

7.1 The combinatorial approach

The fundamental concept in this approach is that of an *anti-chain*.

Definition 7.1 (Anti-chains) A collection $\mathcal{A}$ of sets is known as an *anti-chain* if none of the sets is contained in any other; thus $A \not\subseteq B$ for any distinct $A, B \in \mathcal{A}$.

Anti-chains are sometimes also referred to as *Sperner systems*, especially in older literature.

Lemma 7.2 (LYM inequality) *[240], [246], [385] Let $\mathcal{A}$ be an anti-chain of subsets of a finite set X. Then we have*

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{|X|}{|A|}} \leq 1.$$

Proof We give a probabilistic proof of Bollobás, using Katona's method of random maps. Let $\phi : X \to [1, |X|]$ be a random bijection from X to $[1, |X|]$, chosen uniformly at random among all $|X|!$ such bijections. A simple combinatorial argument shows that

$$\mathbf{P}(\phi(A) = [1, |A|]) = \frac{1}{\binom{|X|}{|A|}}$$

for each $A \in \mathcal{A}$. On the other hand, since none of the A are contained in each other, the events $\phi(A) = [1, |A|]$ are disjoint. Thus, the sum of their probabilities is bounded by 1, which implies the claim. □

From the obvious inequality $\binom{|X|}{|A|} \leq \binom{|X|}{\lfloor |X|/2 \rfloor}$ we immediately conclude

Corollary 7.3 (Sperner's lemma) *[332] Let $\mathcal{A}$ be an anti-chain of subsets of a finite set X. Then $|\mathcal{A}| \leq \binom{|X|}{\lfloor |X|/2 \rfloor}$.*

Note that the bound is clearly optimal, as can be seen by taking $\mathcal{A}$ to be the anti-chain consisting of all subsets of X of cardinality $\lfloor |X|/2 \rfloor$.

We can apply Sperner's lemma to the Littlewood–Offord problem as follows.

Corollary 7.4 *[82] Let $v_1, \ldots, v_n$ be real numbers with $|v_i| \geq 1$ for all i. Let $I = \{x : x_0 - 1 < x < x_0 + 1\}$ be an open interval of length 2. Then the total number of n-tuples $(\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$ with $\epsilon_1 v_1 + \cdots + \epsilon_n v_n \in I$ is at most $\binom{n}{\lfloor n/2 \rfloor}$.*

Proof By reversing the signs of some of the v_i if necessary, we may assume that $v_i > 1$ for all i. Now let $\mathcal{A}$ be the set of all subsets A of $[1, n]$ such that $\sum_{i \in A} v_i - \sum_{i \notin A} v_i \in I$. One can easily verify that $\mathcal{A}$ is an anti-chain, and hence by Sperner's lemma $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$. The claim follows. □

Now let us give a different proof of Sperner's lemma. We need to complement the notion of an anti-chain with that of a chain.

Definition 7.5 (Chains) A *chain* is a sequence of sets $A_1, \ldots, A_m$ such that $A_i \subseteq A_{i+1}$ for all $1 \leq i < m$; we refer to m as the *length* of the chain. We say a chain is *connected* if $|A_{i+1} \backslash A_i| = 1$ for all $1 \leq i < m$. A connected chain in a finite set X is said to be *centered* if $|A_1| + |A_m| = |X|$, or equivalently if $|A_i| = \frac{|X|-m-1}{2} + i$ for all $1 \leq i \leq m$. Note that the length of a centered connected chain has to have the opposite parity as $|X|$.

Lemma 7.6 (Chain decomposition lemma) *[206] Let X be a finite set, and let $2^X = \{A : A \subseteq X\}$ be the power set of X. Then 2^X can be partitioned into disjoint non-empty centered connected chains.*

Proof We induce on $|X|$. The cases $|X| = 0, 1$ are trivial. Now suppose that $|X| > 1$ and the claim has already been proven for all smaller X. Write $X = X' \cup \{x_0\}$ where $|X'| = |X| - 1$. By hypothesis, $2^{X'}$ can be partitioned into disjoint non-empty centered connected chains in X'. For each such chain $A_1, \ldots, A_m$, observe that the chains

$$A_1, \ldots, A_m, A_m \cup \{x_0\}$$

and

$$A_1 \cup \{x_0\}, \ldots, A_{m-1} \cup \{x_0\}$$

are connected centered chains in 2^X, and can be easily be seen to partition 2^X. Note that the chains of the second type may be empty, but they can of course be omitted from the partition without difficulty. The claim follows. □

Every centered connected chain in X has to contain exactly one subset of cardinality $\lfloor X/2 \rfloor$. Thus the total number of chains in Lemma 7.6 is exactly $\binom{|X|}{\lfloor |X|/2 \rfloor}$. More generally, we see the number of centered connected chains of length m given by this lemma is exactly $\binom{|X|}{(|X|-m+1)/2} - \binom{|X|}{(|X|-m-1)/2}$ if m has the opposite parity of $|X|$, and 0 otherwise.

Since an anti-chain can contain at most one element of every chain, we obtain a new proof of Sperner's lemma (compare also with Menger's theorem, Theorem 6.31). In fact, the same argument gives the following generalization.

Proposition 7.7 *[82] Let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be k disjoint anti-chains of subsets of a finite set X. Then*

$$|\mathcal{A}_1| + \cdots + |\mathcal{A}_k| \leq \sum_{i=-\lfloor k/2 \rfloor}^{\lfloor \frac{k-1}{2} \rfloor} \binom{|X|}{\lfloor (|X| + i)/2 \rfloor}.$$

We leave the proof of this proposition as an exercise. We can then extend Corollary 7.4 without difficulty:

Corollary 7.8 (Erdős's Littlewood–Offord inequality) *[82] Let $v_1, \ldots, v_n$ be real numbers with $|v_i| \geq 1$ for all i. Let $I = \{x : x_0 - k < x < x_0 + k\}$ be an open interval of length $2k$ for some integer $k \geq 1$. Then the total number of n-tuples $(\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$ with $\epsilon_1 v_1 + \cdots + \epsilon_n v_n \in I$ is at most $\sum_{i=-\lfloor k/2 \rfloor}^{\lfloor k/2 \rfloor} \binom{n}{\lfloor (n+i)/2 \rfloor}$.*

One can replace the real numbers $\mathbf{R}$ by higher-dimensional spaces, such as the complex numbers $\mathbf{C}$. To do this, we need a product form of Sperner's lemma, as follows.

Lemma 7.9 (Product Sperner lemma) *[206] Let X and Y be finite sets, and let $\mathcal{A}$ be a collection of pairs (A, B) of subsets of X, Y, which are a* product anti-chain *in the sense that there are no distinct pairs $(A, B), (A', B')$ in $\mathcal{A}$ with either $A = A'$ and $B \subsetneq B'$, or $A \subsetneq A'$ and $B = B'$. (To put it another way, for each fixed B, the collection of A for which $(A, B) \in \mathcal{A}$ forms an anti-chain, and vice versa.) Then $|\mathcal{A}| \leq \binom{|X|+|Y|}{\lfloor (|X|+|Y|)/2 \rfloor}$.*

We leave the proof of this lemma as an exercise. As a consequence we have the complex version of Corollary 7.4.

Corollary 7.10 *[206] Let $v_1, \ldots, v_n$ be complex numbers with $|v_i| \geq 1$ for all i. Let $B = \{z : |z - z_0| < 1\}$ be a ball of radius 1. Then the total number of n-tuples $(\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$ with $\epsilon_1 v_1 + \cdots + \epsilon_n v_n \in B$ is at most $\binom{n}{\lfloor n/2 \rfloor}$.*

Proof By randomly rotating the complex plane we may assume that none of the v_i are purely real or purely imaginary. By reversing the signs of some of the v_i if necessary we may assume that $\mathrm{Im}\, v_i > 0$ for all i. Let X be the set of all i with $\mathrm{Re}\, v_i > 0$, and Y be the set of all i with $\mathrm{Re}\, v_i < 0$; thus $X \cup Y = [1, n]$. Now let $\mathcal{A}$ be the set of all pairs (A, B) of sets $A \subset X$, $B \subset Y$ such that $\sum_{i \in A \cup B} v_i - \sum_{i \notin A \cup B} v_i \in I$. One can easily verify that $\mathcal{A}$ is a product anti-chain in the sense of Lemma 7.9, and the claim follows. $\square$

In fact one has the analogous claim in general dimension, by a more sophisticated version of this argument; see [207].

This is only the tip of the iceberg concerning extremal combinatorics results of this type; see for instance [32] for a much more detailed treatment of these topics. Variants of this approach have also been successfully applied in cyclic groups; see [163].

Exercises

7.1.1 (Set-pair estimate)[31] Let $A_1, \ldots, A_m, B_1, \ldots, B_m$ be finite sets such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. Show that

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

Note that this includes Lemma 7.3 as a special case (where $B_i := X \backslash A_i$).

7.1.2 (Erdős–Ko–Rado theorem) [94] Let $A_1, \ldots, A_m$ be an anti-chain in $\mathbf{Z}_N$ such that any two A_i, A_j intersect (thus $A_i \cap A_j \neq \emptyset$ for all i, j), and $|A_i| \leq k$ for all i and some $k \leq N/2$. Show that $m \leq \binom{N-1}{k-1}$, and show that this bound is sharp. (Hint: first show that for any bijection $\phi : \mathbf{Z}_N \to \mathbf{Z}_N$, at most k of the sets $\phi(A_i)$ can be an interval of the form $[a + 1, a + |A_i|]$ for some $a \in \mathbf{Z}_N$; this elegant argument is due to Katona [196].)

7.1.3 Prove Proposition 7.7. (Hint: for any chain of length m, observe that at most $\min(m, k)$ elements of this chain can lie in $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$. Now count how many chains there are of a given length in Lemma 7.6.)

7.1.4 Prove Proposition 7.9. (Hint: if $A_1, \ldots, A_m$ is a connected chain in X, and $B_1, \ldots, B_n$ is a connected chain in Y, show that there are at most $\min(m, n)$ pairs of the form (A_i, B_j) in $\mathcal{A}$. Alternatively, decompose 2^Y into chains $B_1, \ldots, B_n$, and for each such chain apply Proposition 7.7.)