where $\epsilon_1, \ldots, \epsilon_{2k} \in \{-1, +1\}$ and $i_1, \ldots, i_{2k} \in [1, n]$. In particular, if the v_j are all distinct and take values in a set S, then we have

$$\mathbf{P}\big(X_{\mathbf{v}}^{(\mu)} = x\big) \leq O\big(\mu^{-1/2} n^{-9/2} \mathbf{E}(S, S)\big).$$

Thus $X_{\mathbf{v}}^{(\mu)}$ can only concentrate significantly when the support has substantial additive energy. Explain heuristically why this result is related to the $\mu = 2k/n$ case of Lemma 7.14.

7.3 The Esséen concentration inequality

In several applications, we are not interested in the probability that a random walk $X_{\mathbf{v}}^{(\mu)}$ ends up in a specified point, but rather in a region of space such as a cube. In some "discrete" cases (e.g. when the $v_1, \ldots, v_n$ live in a lattice) one can simply use the union bound to pass from the former to the latter, but this is not always the best approach. One useful tool for dealing with concentration in general is a simple concentration inequality of Esséen.

Lemma 7.17 (Esséen concentration inequality) *[101] Let X be a random variable taking a finite number of values in $\mathbf{R}^d$. Let $x_0 \in \mathbf{R}^d$, and let $R, \varepsilon > 0$. Then*

$$\sup_{x_0 \in \mathbf{R}^d} \mathbf{P}(|X - x_0| \leq R) = O\left(\frac{R}{\sqrt{d}} + \frac{\sqrt{d}}{\varepsilon}\right)^d \int_{\xi \in \mathbf{R}^d : |\xi| < \varepsilon} |\mathbf{E}(e(\xi \cdot X))| \, d\xi.$$

Here $e(x) := \exp(2\pi i x)$, $\xi \cdot X$ denotes the usual inner product on $\mathbf{R}^d$, and $|\xi|$ denotes the usual magnitude.

Proof By rescaling X and R by ε we may take $\varepsilon = \sqrt{d}$. A simple covering argument (using for instance Corollary 3.15) then shows that it suffices to show that

$$\mathbf{P}(|X - x_0| \leq c\sqrt{d}) \leq O(1)^d \int_{\xi \in \mathbf{R}^d : |\xi| < \sqrt{d}} |\mathbf{E}(e(\xi \cdot X))| \, d\xi$$

for all $x_0 \in \mathbf{R}$ and some small absolute constant $c > 0$. By translating X by x_0 (which does not affect the right-hand side) we may take $x_0 = 0$. Now from the standard Gaussian integral identity

$$\int_{\xi \in \mathbf{R}^d} e^{-\pi C |\xi|^2} e(\xi \cdot X) \, d\xi = C^{-d/2} e^{-\pi |X|^2 / 2}$$

for any $C > 0$, we see that

$$\left| \int_{\xi \in \mathbf{R}^d : |\xi| < \sqrt{d}/2} e^{-\pi C |\xi|^2} e(\xi \cdot X) \, d\xi \right| = \Omega(1)^d \tag{7.4}$$

whenever $|X| \leq c\sqrt{d}$, if c is chosen sufficiently small and C chosen sufficiently large. Squaring this we obtain

$$\int_{\xi \in \mathbf{R}^d : |\xi| < \sqrt{d}} e(\xi \cdot X) w(\xi)\, d\xi = \Omega(1)^d \mathbf{I}(|X| \leq c\sqrt{d})$$

where $w(\xi) := \int_{|\xi_1|, |\xi - \xi_1| < \sqrt{d/2}} e^{-\pi C |\xi_1|^2} e^{-\pi C |\xi - \xi_1|^2}$. Taking expectations of both sides we obtain

$$\int_{\xi \in \mathbf{R}^d : |\xi| < \sqrt{d}} |\mathbf{E}(e(\xi \cdot X))| w(\xi)\, d\xi \geq \Omega(1)^d \mathbf{P}(|X| \leq c\sqrt{d}).$$

From (3.8) we see that $w(\xi) = O(1)^d$, and the claim follows. □

Applying this in particular to the random variable $X_{\mathbf{v}}^{(\mu)}$ for some $\mathbf{v} = (v_1, \ldots, v_n)$ and $0 \leq \mu \leq 1$ we obtain the following analog of Lemma 7.11:

$$\mathbf{P}\left(\left|X_{\mathbf{v}}^{(\mu)} - x_0\right| \leq R\right) = O\left(\frac{\sqrt{d}}{\varepsilon} + \frac{R}{\sqrt{d}}\right)^d \int_{\xi \in \mathbf{R}^d : |\xi| < \varepsilon} \prod_{j=1}^{n} |1 - \mu + \mu \cos(2\pi \xi \cdot v_j)|\, d\xi. \tag{7.5}$$

As an application we present a higher-dimensional analog of Corollary 7.10, but with the loss of a dimension-dependent constant.

Proposition 7.18 *[207], [167] Let $0 < \mu \leq 1$, and suppose n is sufficiently large depending on μ. Let $v_1, \ldots, v_n$ be elements of $\mathbf{R}^d$ with $|v_i| \geq 1$ for all i. Then for any $x_0 \in \mathbf{R}^d$, we have*

$$\mathbf{P}\left(\left|X_{\mathbf{v}}^{(\mu)} - x_0\right| \leq k\right) \leq O(1)^d \frac{k}{\sqrt{\mu n}}$$

for all $k \geq 1$.

It is worth noting that the right-hand side grows only linearly in k, instead of the k^d type growth that one might naively expect. This is a reflection of the heuristic that the random variable $X_{\mathbf{v}}^{(\mu)}$ tends to concentrate the strongest on one-dimensional spaces (cf. Exercise 7.2.3).

Proof In view of (7.5) (with $R = k$ and $\varepsilon = 1/k$), it suffices to show that

$$\int_{\xi \in \mathbf{R}^d : |\xi| \leq 1/k} \prod_{j=1}^{n} |1 - \mu + \mu \cos(2\pi \xi \cdot v_j)|\, d\xi = O\left(\frac{1}{k\sqrt{d}}\right)^d \frac{k}{\sqrt{\mu n}}.$$

Applying Hölder's inequality, we reduce to showing that

$$\int_{\xi \in \mathbf{R}^d : |\xi| \leq 1/k} |1 - \mu + \mu \cos(2\pi \xi \cdot v_j)|^n\, d\xi = O\left(\frac{1}{k\sqrt{d}}\right)^d \frac{k}{\sqrt{\mu n}}$$