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## Incidence geometry

Incidence geometry deals with the incidences among basic geometrical objects such as points, lines and spheres. One can obtain useful and non-trivial information on these incidences by the classical combinatorial technique of *double-counting* the number of a certain type of configuration of incidences in two different ways.

In many situations, tools from incidence geometry, combined with a clever double counting argument provide a simple, yet powerful, approach to hard problems. The goal of this chapter is to demonstrate several such applications, including several in additive combinatorics.

The material is organized as follows. We start with a result on the *crossing number* of graphs, which has a topological flavor. Next, we use this result to give simple proof of the famous *Szemerédi–Trotter theorem* concerning point-line incidences. In the next two sections, we use this theorem to prove several bounds on the Erdős–Szemerédi sum-product problem and reprove Andrew’s theorem on the number of lattice points in a convex polygon. Next, we introduce the method of *cell decomposition* and use it to treat Erdős distinct distances problem in  $\mathbf{R}^d$ . Finally, we discuss a variant of Erdős–Szemerédi sum-product problem for complex numbers.

### 8.1 The crossing number of a graph

In this chapter, a *point* refers to a point in the plane  $\mathbf{R}^2$ , and a *line* refers to a line in  $\mathbf{R}^2$ , unless otherwise specified. By a *curve*, we refer to the image of a continuous injective embedding<sup>1</sup> of a compact interval  $[0, 1]$  into  $\mathbf{R}^2$ .

<sup>1</sup> In applications one deals with very explicit curves such as circular arcs or straight lines, and so we could restrict the class of curves to these sorts of objects if desired. In this way one does not need to invoke any difficult results from topology such as the Jordan curve theorem (which is implicit in our application of Euler’s formula).

Consider a graph  $G = G(V, E)$ ; recall we assume our graphs  $G$  to be undirected and have no loops or repeated edges. A *drawing* of  $G$  is any representation of  $G$  in the plane  $\mathbf{R}^2$  by identifying each vertex in  $V$  with a distinct point, and each edge  $(u, v)$  in  $E$  with a curve in  $\mathbf{R}^2$  connecting  $u$  and  $v$ . The *crossing number* of such a drawing is the number of pairs of edges with no common endpoints, where the corresponding curves intersect each other. The *crossing number* of  $G$  is the minimum number of crossings in a drawing. Here and later, we denote this parameter by  $\text{cross}(G)$ .

It is expected that if  $G$  has many edges, then its crossing number is large. The following theorem, which confirmed this intuition, was proved by Ajtai, Chvátal, Newborn and Szemerédi [1], and, independently, by Leighton [224].

**Theorem 8.1** *Let  $G = G(V, E)$  be a graph with  $|E| \geq 4|V|$ . Then  $\text{cross}(G) \geq \frac{|E|^3}{64|V|^2}$ .*

*Proof* A *planar graph* is a graph whose crossing number is zero. It is well known (and can be easily proved using Euler's formula) that a planar graph  $G = G(V, E)$  has at most  $3|V|$  edges (in fact it has at most  $3|V| - 6$  if  $|V| \geq 3$ ). Now observe that any graph  $G$  can be made planar by removing at most  $\text{cross}(G)$  edges (one for each crossing that occurs in an optimal drawing of  $G$ ). Combining these two facts we obtain the preliminary inequality

$$\text{cross}(G) \geq |E| - 3|V| \tag{8.1}$$

for an arbitrary graph  $G(V, E)$ .

This bound is, of course, much weaker than what we want to prove. However it is possible to amplify (8.1) substantially via the first moment method as follows.

Fix  $G = G(V, E)$  with  $|E| \geq 4|V|$ , and let  $0 < p \leq 1$  be a parameter to be chosen later. Let  $V'$  be a random subset of  $V$ , chosen so that the events  $v \in V'$  are independent with probability  $p$ . Let  $G' = G'(V', E')$  be the induced subgraph of  $G$  spanned by  $V'$ . Applying (8.1) to  $G'$  and then taking expectations, we see from linearity of expectation that

$$\mathbf{E}(\text{cross}(G')) \geq \mathbf{E}(|E'|) - 3\mathbf{E}(|V'|).$$

Further application of linearity of expectation shows that

$$\mathbf{E}(|V'|) = p|V|; \quad \mathbf{E}(|E'|) = p^2|E|,$$

since each vertex has probability  $p$  of being included in  $V'$ , and each edge has probability  $p^2$  of being included in  $E'$ . Now consider a drawing of  $G$  with exactly  $\text{cross}(G)$  crossings. Each crossing involves four vertices of  $V$  and thus has a probability  $p^4$  of surviving when we pass to  $G'$ . Using linearity of expectation one

last time we conclude

$$\mathbf{E}(\text{cross}(G')) \leq p^4 \text{cross}(G);$$

we have inequality rather than equality since the drawing of  $G'$  constructed here may not have the minimal number of crossings. Putting all this together we have

$$\text{cross}(G) \geq p^{-3}|E| - 3p^{-2}|V|.$$

The claim then follows by setting  $p := 4|V|/|E|$ .  $\square$

**Remark 8.2** One can improve the bound on  $\text{cross}(G)$  slightly by optimizing  $p$ . To obtain a more significant improvement, one needs additional arguments. The current best bound is due to Pach and Tóth [271].

### Exercises

- 8.1.1 Let  $G(V, E)$  be a planar graph with no loops or multiple edges. Using Euler's formula  $V - E + F = 2$ , show that  $|E| \leq \max(3|V| - 6, 1)$ . Show that this bound  $\max(3|V| - 6, 1)$  is best possible.
- 8.1.2 Show that a planar graph has a vertex of degree at most 5. Use this fact and induction to prove that a planar graph is vertex-colorable by 6 colors, where of course we require adjacent vertices to have distinct colors. Without using the four-color theorem, refine the argument to show that in fact every planar graph is vertex-colorable by 5 colors. (Hint: given any two colors, say red and green, one can swap all the red and green colors in a single red-green connected component without difficulty. Now given four colors red, blue, green, white adjacent in that order around a single uncolored vertex  $v$ , it cannot simultaneously be true that the red and green vertices lie in the same red-green connected component, and the blue and white vertices lie in the same red-white connected component.)
- 8.1.3 Show that for any  $n, e \geq 1$  with  $e \geq 4n$  that there exists a graph  $G = G(V, E)$  with  $n$  vertices and  $e$  edges such that  $\text{cross}(G) = \Theta(e^3/n^2)$ , and so the crossing number inequality cannot be improved except for constants. (Hint: There are many ways to generate an example. One is to connect adjacent and nearly-adjacent points on the unit circle. Another is to use Exercise 8.2.2.)
- 8.1.4 Show that for any graph  $G = G(V, E)$ , we have  $|E| = O(|V| + |V|^{2/3} \text{cross}(G)^{1/3})$ .
- 8.1.5 [342] Let  $m \geq 1$  be an integer, and let  $G = G(V, E)$  be a multigraph with maximum edge multiplicity  $m$ , thus each pair of vertices are allowed to be connected by up to  $m$  edges. Define the crossing number of a multigraph in the obvious manner. Show that if